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EVOLUTION OF HELE SHAW INTERFACE FOR SMALL SURFACE TENSION

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ABSTRACT

For the time evolution of a Hele-Shaw interface described by a conformal map $z(\zeta, t)$ that maps a unit circle (or a semi-circle) in the ζ plane into the viscous fluid flow region in the physical z-plane, we present results on the motion of singularities outside the unit circle.

For zero surface tension, we extend earlier results to show that for any initial condition, each singularity of $z(\zeta, t)$ present initially in $|\zeta| > 1$ continually approaches the interfacial boundary $|\zeta| = 1$ without any change of form. However, depending on the singularity type, it may or may not impinge $|\zeta| = 1$ in finite time. Under some assumptions, we give analytical evidence to suggest that the ill-posed problem in the physical domain $|\zeta| \leq 1$ can be imbedded in a well-posed problem in $|\zeta| \geq 1$. We present a numerical scheme to calculate such solutions.

For each initial singularity of certain type, which in the absence of surface tension would have merely moved to a new location $\zeta_s(t)$ at time t from an initial $\zeta_s(0)$, we find an immediate transformation of the singularity structure for nonzero surface tension $\mathcal{B}$; however, for $0 < \mathcal{B} << 1$, surface tension effects on this singularity are limited to a small 'inner' neighborhood of $\zeta_s(t)$ when $t << \frac{1}{\mathcal{B}}$. Outside the inner-region but for $|\zeta - \zeta_s(t)| << 1$, the singular behavior of z^0, the zero surface tension solution still persists for $z(\zeta, t)$. On the other hand, for each initial zero of z_ζ, which for surface tension $\mathcal{B} = 0$ remains a zero of z^0_ζ at a location $\zeta_0(t)$ different from $\zeta_0(0)$, surface tension effects spawns new singularities that move away from $\zeta_0(t)$ and approach the physical domain $|\zeta| = 1$.

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1. Introduction

The problem of a less viscous fluid displacing a more viscous fluid in a Hele-Shaw cell has received considerable attention over the last few years since the original work by Saffman & Taylor (1958). Reviews by Saffman (1986), Bensimon et al (1986), Homsy (1986) summarize the state of affairs as of five years back. Since then, there has been considerable work that have been reviewed from different perspectives by Pelce (1988), Kessler & Levine (1988), Howison (1991) and Tanveer (1991). This intense activity related to a relatively simple experiment has been motivated by the mathematical analogies of the Hele-Shaw to viscous displacement in a porous medium, growth of a needle crystal in an undercooled melt and the morphological instability and resulting features in directional solidification. Indeed, many of the generic results for these apparently diverse areas were first discovered in a Hele-Shaw flow. In addition to the relative simplicity of the equations, the flow in this geometry is accessible to experiments making it possible to compare theory with experiment.

Most of the work to date has been on steady states and their linear stability (see recent reviews cited above for extensive bibliography). In an actual experiment (Saffman & Taylor (1958), Tabeling et al (1986), Maxworthy (1987), Arneodo et al (1989) the interface does not evolve into a steady state when surface tension effects are very small. Indeed, through a sequence of instabilities, a highly convoluted interfacial pattern forms that appears to be fractal (Arneodo et al (1989)) over some range of length scales. Further, the precise interfacial shape appears to be extremely sensitive to initial conditions though the overall statistics of the observed pattern is not. The theoretical understanding of these features is lacking to a great degree. This paper, we hope, is an important step in that direction.

The time evolution problem is studied in this paper is for a highly idealized boundary condition, originally developed by by Mclean & Saffman (1980) (we call it the MS boundary conditions), where important three dimensional effects observed in the experiment (Tabeling et al (1986)) are ignored. For steady flow, realistic two dimensional boundary conditions incorporating three dimensional thin film effects were discussed by Saffman (1982) and later derived by Park & Homsy (1985) and Reinelt (1987) (we call it the SPHR boundary conditions). Numerical (Schwartz & DeGregoria (1987), Reinelt (1987), Sarkar & Jasnow (1987)) and analytical calculations (Tanveer, 1990) show important quantitative differences in solutions corresponding to the MS and SPHR boundary conditions. Over some range of experimental control parameters, qualitative differences also exist though in other cases the qualitative conclusions are about the same. Further, analytical calculations on steady state finger (Tanveer, 1990) suggest that the mathematical structure, in a broad sense, are similar in these two cases despite the relative complexity of the SPHR conditions.

Here, we proceed with the belief that for the time evolving flow, the solutions corresponding to the much simpler MS boundary conditions are relevant, atleast qualitatively, over some range of control parameters, as is true for the steady state.

For the rectilinear geometry, we take $z(\zeta,t)$ to be the conformal map that maps the semi-circle in Fig. 1 into the physical flow domain (Fig. 2) in the z plane such that $\zeta = 0$ corresponds to $z = +\infty$ and $\zeta = \pm 1$ correspond to points A and B (see Fig. 2), where the interface meets the channel side walls. It is clear that one can decompose

$$z(\zeta,\ t) \ = \ -\frac{2}{\pi}\ ln\ \zeta \ + \ i \ + \ f(\zeta,\ t) \tag{1.1}$$

where f is analytic in the unit semi-circle with

$$Im\ f \ = \ 0 \tag{1.2}$$

on the real diameter of the unit circle corresponding to the geometric condition that the walls of the channel correspond to the real diameter of the unit semi-circle. We will assume f is continuous up to the real diameter including $\zeta = 0$. Further, we will assume that the shape of the extended finger obtained by reflection about each of the two side walls is smooth implying that z and hence f is analytic and $z_\zeta \neq 0$ on the semi-circular arc $|\zeta| = 1$. ¿From Schwarz reflection principle, f is analytic and $z_\zeta \neq 0$ for $|\zeta| \leq 1$. We decompose the complex velocity potential:

$$W(\zeta,\ t) \ = \ -\frac{2}{\pi}\ ln\ \zeta \ + \ i \ + \ \omega(\zeta,\ t) \tag{1.3}$$

where the fluid velocity at infinity is assumed to be unity without any loss of generality (equivalent to appropriate nondimensionalization). It is clear that the condition of no flow through the walls imply

$$Im\ \omega \ = \ 0 \tag{1.4}$$

on the real diameter (-1, 1) of the unit semi-circle. As is physically reasonable, ω will be assumed to be continuous up to the real diameter including $\zeta = 0$. Further, it will be assumed that ω is analytic on the semicircle corresponding to the assumption of smooth flow at the interface. ¿From Schwarz reflection principle, (1.4) implies that ω is analytic in $|\zeta| \leq 1$.

Second, for the radial geometry, following Howison (1985,1986ab), the conformal map $z(\zeta,\ t)$ that maps the interior of the unit circle in the ζ plane into the physical flow domain (Fig. 1) such that $\zeta = 0$ is mapped to ∞ can be decomposed as

$$z(\zeta,\ t) \ = \ \frac{a(t)}{\zeta} \ + \ k(\zeta,\ t) \tag{1.5}$$

where the remaining degree of freedom in the Riemann mapping theorem is used by requiring $a(t)$ to be real and positive. We will assume that the interface is smooth and so $k(\zeta,\ t)$ is analytic and $z_\zeta \neq 0$ everywhere in $|\zeta| \leq 1$. $k(\zeta, t)$ can be expressed as as power series convergent for $|\zeta| \leq 1$:

$$k(\zeta, t) = \sum_{n=0}^{\infty} k_n(t)\, \zeta^n \tag{1.6}$$

Further, the complex velocity potential $W(\zeta,\ t)$ can be decomposed as:

$$W(\zeta,\ t) = -\frac{2}{\pi}\, ln\, \zeta + \omega(\zeta,\ t) \tag{1.7}$$

where the injection rate $Q = 4$ without any loss of generality. This choice is made for convenience of presentation so that (1.3) and (1.7) have the same form. For smooth flow at the interface, $\omega(\zeta,\ t)$ can be assumed to be analytic in $|\zeta| \leq 1$ and has a convergent series representation in that domain:

$$\omega(\zeta, t) = \sum_{n=0}^{\infty} \omega_n(t)\, \zeta^n \tag{1.8}$$

Note that unlike the channel geometry, ω and k do not have vanishing imaginary part on the real axis within the unit circle except if they are so initially (corresponds to certain conditions of symmetry of initial conditions).

In either Hele-Shaw cell geometries, the boundary condition (Mclean-Saffman, 1980) on the finger boundary that the difference of pressure on the two sides is balanced by surface tension times curvature implies that on $|\zeta| = 1$

$$Re\ \omega = -\frac{\mathcal{B}}{|z_\zeta|}\, Re \left[1 + \zeta\, \frac{z_{\zeta\zeta}}{z_\zeta} \right] \tag{1.9}$$

where for the channel geometry, $\mathcal{B} \equiv \frac{b^2 T}{12 \mu V a^2}$, where T is the surface tension, b is the gap width and $2a$ is the cell width, μ the viscosity of the more viscous fluid, V the velocity of fluid at ∞. For the radial Hele-Shaw geometry, we define $\mathcal{B} \equiv \frac{b^2 T}{3 \mu Q a}$, where Q is the injection rate and $\pi\, a^2$ is the initial area of the blob of less viscous fluid. The viscosity of the less viscous fluid and the thin film effects have been neglected here for the sake of simplicity. The kinematic boundary condition that corresponds to no fluid flow through the interface (see Saffman (1959) or Richardson (1972) for details) implies

$$Re \left[\frac{\zeta\, W_\zeta}{|z_\zeta|^2} - \frac{z_t}{\zeta z_\zeta} \right] = 0 \tag{1.10}$$

on $|\zeta| = 1$. Note that for the channel geometry, (1.9) and (1.10) hold on the lower half unit semi-circular boundary as well. The mathematical problem, therefore, is to determine analytic functions ω and f (or k for the radial geometry) and therefore $W(\zeta, t)$ and $z(\zeta, t)$ satisfying (1.9) and (1.10) on the circular arc, when $z(\zeta, 0)$ and $W(\zeta, 0)$ are appropriately specified.

Almost all the work to date on this initial value problem* belong to either of two categories: numerical computations for $\mathcal{B}$ not too small or analytical work for $\mathcal{B} = 0$.

For $\mathcal{B} \neq 0$, the numerical computations by Trygvasson & Aref (1983), DeGregoria & Schwartz (1986), Bensimon (1986) and Meiburg & Homsy (1988) have shed some light on the process of finger competition that eventually leads to a single steadily moving finger. However, when surface tension parameter is smaller than some critical value, the finger is itself unstable. The calculations of linear stability by Kessler & Levine (1985, 1986) and Tanveer (1987b) suggests that there exists a linearly stable branch for any nonzero surface tension. Based on numerical experimentation, Bensimon (1986) concluded that observed instability in the simulation is a nonlinear mechanism where the threshold amplitude for destabilization decreases with surface tension. This is plausible, as a discrete set of steady fingers (VandenBroeck, 1983) coalesce as $\mathcal{B} \rightarrow 0$ and all but one of these branches are unstable (Kessler & Levine (1987), Tanveer (1987b), Bensimon et al (1987)). Despite the understanding on the onset of various instabilities, a full understanding of the complicated time dependent pattern beyond the onset stage is difficult using conventional simulation with boundary integral methods since widely disparate scales, both in space and time, need to be resolved accurately as $\mathcal{B}$ is made progressively smaller.

For $\mathcal{B} = 0$, it is clear that the initial value problem posed in (1.9) and (1.10) simplify considerably since $\omega = 0$ in this case and therefore $W(\zeta, t)$ is known (from (1.7)). Gustaffson (1984, 1987) has rigorously proved the existence of solution for finite time for analytic initial data on the real axis (the physical domain in his formulation). Earlier, Galin (1945) and Polubarinova-Kochina (1945) considered the mathematically identical problem of Darcy model for ground water flows and devised analytical techniques for obtaining exact solutions. These were apparently well known in the Russian literature (see Howison (1991) & Hohlov (1990)). Exact solutions due to Saffman (1959), Howison (1985, 1986ab) and Shraiman & Bensimon (1985) can be seen as applications of these techniques though results appear to have been obtained without knowledge of the Russian work. Howison (1990) summarizes the relation between the different techniques. The Saffman-Howison-Shraiman-Bensimon (SHSB) exact solutions are limited to initial conditions for which $z_\zeta(\zeta, 0)$ is either a polynomial or a superposition of poles. In these cases, the number

* Other authors have used differing but equivalent formulation

of poles is known to be preserved, as is the number of zeros; only their locations change. Within this class of solutions, it is known that poles never hit the physical domain boundary $|\zeta| = 1$ in finite time but zeros may or may not. In the case when a zero hits $|\zeta| = 1$, a zero angled cusp protruding into the viscous fluid forms at the physical interface. The mathematical solution does not make physical sense *beyond that time*. Howison (1986a) was further able to show that that one can start with two different initial conditions that are arbitrarily close in the physical domain $|\zeta| \leq 1$ (in any Sobolev norm), yet after a finite time, the physical boundary develops a cusp in one case while it remains regular in the other. In this sense, it is known that the zero surface tension evolution problem is ill posed. Richardson (1972) showed certain invariance properties of the Schwartz function of a domain for the related problem of a more viscous fluid displacing a less viscous one. In our formulation, Richardson's result implies that singularities of $z(\zeta, t)$ in the finite ζ plane other than that at $\zeta = 0$, arise due motion of an initial singularity of the same form (i.e. for instance, a branch point of one half power remains so at later times) of $z(\zeta, 0)$ in $|\zeta| > 1$. However, no explicit solution are known if $z_\zeta(\zeta, 0)$ contains branch point. Richardson's suggestion about approximating them by a polynomial is invalid in our case as we shall see that the later time evolution depend critically on the initial singularity structure in $|\zeta| > 1$. The only known closed form solutions with branch points are the self-similar *solutions in time for a radial or wedge geometry* recently found by Thome et al (1988), Ben Amar (1991) and Tu (1991). Further, due to the ill-posedness of the problem in the physical domain $|\zeta| \leq 1$, conventional numerical simulation using boundary integral methods for instance is not possible due to uncontrolled growth of round off errors, unless one uses some artificial dissipation or a filtering procedure (see Krasny (1986) for applications to another ill-posed interfacial problem). Such procedures appear to be computationally expensive and may not be practical when the interface becomes highly convoluted as is the case in the interesting stages of Hele-Shaw evolution.

For nonzero $\mathcal{B}$, it is expected that the initial value problem in the physical domain will be well-posed. However, the problem is considerably more intractable analytically, because of complications due to the unwieldly curvature term on the right hand side of (1.9). As stated earlier, numerical computations in the physical domain become rather difficult in the dynamically interesting case of small $\mathcal{B}$. Duchon & Robert (1984), while working on a mathematically equivalent problem, have proved the existence of solutions for short times. However, it is not known if solutions exists for all times for any nonzero $\mathcal{B}$. Physically, one might expect this to be the case, though this remains to be shown. Studying the problem perturbatively about any known solution at $\mathcal{B} = 0$ is fraught with difficulties. If $\mathcal{S}_0(t)$ denotes the solution operator that maps the initial condition $z(\zeta, 0)$ in

$|\zeta| \leq 1$ into the $\mathcal{B} = 0$ solution $z^0(\zeta, t)$ in $|\zeta| \leq 1$, ill posedness immediately implies that the operator $\mathcal{S}_0(t)$ is unbounded. For this reason, there is no reason to expect under a small perturbation caused by a small nonzero $\mathcal{B}$, the corresponding solution operator $\mathcal{S}(t)$ will be close to $\mathcal{S}_0(t)$. Any physical argument to the contrary is unacceptable as a physical flow always corresponds to nonzero $\mathcal{B}$ and one cannot use physical argument to extrapolate the $\mathcal{B} = 0$ result. Despite this, in an effort to make some headway, analytical efforts have been made by Lacey et al (1988) by assuming that for small surface tension, surface tension effects are important only when the interface curvature, as predicted by z^0, is large. As the authors themselves point out, this assumes that the actual interface curvature is not very different from what is predicted by z^0, the zero surface tension solution. Recently, Dai et al (1991) carry out a numerical simulation of the initial value problem in the physical domain for the small nonzero $\mathcal{B}$ in the radial geometry when z_ζ is either a cubic polynomial or a combination of three poles with a three fold rotational symmetry and deduce the nature of singularities in the unphysical domain using a Domb-Sykes plot. For such initial conditions, exact solutions are known within the Saffman-Howison-Bensimon-Shraiman family when $\mathcal{B} = 0$. Comparison of Dai et al (1991) calculations with these suggest that initial poles of z_ζ remain unaffected. With polynomial initial conditions, it is found that an initial zero splits into what they term as 'two pole-like singularities'. In this case, it is found that difference with the exact $\mathcal{B} = 0$ solution becomes significant even when the interface curvature based on $z^0(\zeta, t)$ is $<< \frac{1}{\mathcal{B}}$. This appears to suggest that the Lacey et al (1988) hypothesis may not be correct. Recently, Constantin & Kadanoff (1990, 1991) have derived local partial differential equations for $z(\zeta, t)$ in the region $|\zeta| > 1$ for the radial geometry by analytically continuing (1.9) and (1.10) for the radial geometry, assuming that the singularities are very far off from the physical plane so that the physical interface is nearly circular. Their localization procedure is similar in spirit to the equations derived by Caflisch, Orellana & Siegel (1990) for the zero surface tension case. Constantin & Kadanoff (1991) and Howison (1991) have also found special similarity solutions that show the immediate transformation of an initial singularity or a zero of z_ζ into possible -4/3 singularities; however their relevance to the overall physical dynamics has not been addressed. Constantin & Kadanoff (1990) also prove the existence of solutions to these localized equations in an unphysical domain that shrinks in time like $t^{-1/2}$. Since the physical domain is $|\zeta| \leq 1$ (in their formulation, $|\zeta| \geq 1$), their results do not guarantee existence of solutions for all times even with surface tension. However, their equations become invalid when the actual interfacial shape deviates significantly from a circle, long before a singularity actually can come close to $|\zeta| = 1$. Thus the behavior of their solution need not reflect the behavior of actual solutions that satisfy (1.9) and (1.10).

We like to understand the Hele-Shaw dynamics for small nonzero $\mathcal{B}$ perturbatively by exploiting the simplicity of the equations when $\mathcal{B} = 0$. There are two major hurdles in accomplishing this. First, the information on the $\mathcal{B} = 0$ problem is not complete. We do not have exact solutions for a general initial condition. Nor is it known how to compute such solutions effectively. Second, $\mathcal{B} = 0$ problem is ill posed in the physical domain $|\zeta| \leq 1$, as demonstrated by Howison (1985, 1986ab). Thus, as discussed before, one cannot have any confidence that a zero surface tension solution $z^0(\zeta, t)$ in the physical domain $|\zeta| \leq 1$ is necessarily relevant as $\mathcal{B} \to 0$. In this paper, we address these two problems.

First, we show that the analytically continued equation for $\mathcal{B} = 0$ has the feature that all information in $|\zeta| > 1$, flows inwards towards the interface boundary $|\zeta| = 1$. In particular, for arbitrary initial conditions, singularities present in $|\zeta| > 1$ must always approach the physical domain boundary $|\zeta| = 1$, though it need not actually impinge it in finite time. Assuming that that the speed of a zero of $z_\zeta(\zeta, t)$ is bounded for certain classes of initial conditions, we present analytical arguments to suggest that the initial value problem in $|\zeta| \geq 1$ obtained by solving the analytically continued equations with analytically continued initial condition is well posed in the unphysical domain for a finite time. We also argue that the solution obtained in this way for $t > 0$ is the analytical continuation of the physical solution in $|\zeta| \leq 1$ across the unit circle. Using these results, we suggest a numerical method to find solutions for arbitrary initial conditions, including ones where z_ζ has a set of branch points of specific types. Because of the well-posedness of the underlying formulation in $|\zeta| > 1$, the computation can be performed without growth of round off errors, a problem that plagues computation of solutions to ill-posed problems. Recently, G.R. Baker (private communication) has successfully implemented this scheme. We then present analytical evidence that a singularity $\zeta_s(t)$, where $z_\zeta \sim B_0(t)(\zeta - \zeta_s(t))^{-\beta}$, does not impinge the physical domain when $\beta \geq \frac{1}{2}$. When $0 < \beta < \frac{1}{2}$, our evidence suggests that the singularity impinges $|\zeta| = 1$ in finite time. Under some assumptions that we are unable to check, the same appears to be true for $\beta < 0$. Our results also suggests that the ill-posedness in the physical domain for $\mathcal{B} = 0$ is actually in the sense of Hadamard.

Having found a well posed formulation in the domain $|\zeta| \geq 1$ allows us to address the second serious hurdle mentioned above. In this extended domain, we can expect that the solution $z(\zeta, t)$ for nonzero is close in some sense to $z^0(\zeta, t)$ and therefore can be studied perturbatively. Indeed, a minor modification of the numerical method suggested for solving the zero surface tension solution $z^0(\zeta, t)$ can, in principle, be used to find each term of the regular perturbation expansion in $|\zeta| \geq 1$:

$$z(\zeta, t) = z^0(\zeta, t) + \mathcal{B}\, z^1(\zeta, t) + \ldots \tag{1.11}$$

Examination of the analytically continued equation in $|\zeta| > 1$ shows immediately that the

regular perturbation expansion expansion will break down near each point $\zeta_0(t)$ where $z^0_\zeta = 0$ and singularities that include branch points $\zeta_s(t)$ near which $z^0_\zeta \sim B_0(t)(\zeta - \zeta_s(t))^{-\beta}$ with $\beta < 2$. Even for $\mathcal{B}$ independent initial conditions, the perturbation expansion (1.11) becomes invalid at any singular point ζ_p where $z_\zeta \sim \tilde{A}(t)[\zeta - \zeta_p(t)]^{-4/3}$, with $\tilde{A}(0) = 0$ and $\tilde{A}(t)$ scaling as some positive power of $\mathcal{B}$. As we shall see later, such singular points are created at the initial instant of time by surface tension effects. When, the perturbation expansion (1.1) breaks down, it becomes necessary to consider the inner equation obtained by appropriately rescaling the dependent and independent variables.

With certain assumptions, we find that for initial conditions that are independent of $\mathcal{B}$, appropriate inner-outer matching can be carried out for $0 < \beta < 2$. We find that for a branch point or a simple pole singularity of $z_\zeta(\zeta, 0)$ where $z_\zeta(\zeta, 0) \sim B_0(0)\,(\zeta - \zeta_s(0))^{-\beta}$, for a nonzero $\mathcal{B}$, the singularity is immediately transformed. However, for $0 < \mathcal{B} << 1$ and $t << \frac{1}{\mathcal{B}}$, the location of the newly created singularities is within a small inner neighborhood of $\zeta_s(t)$ (singularity of z^0), where $|\zeta - \zeta_s(t)| = O(\mathcal{B}^{\frac{2}{3(2-\beta)}})$. The singular behavior $(\zeta - \zeta_s(t))^{-\beta}$ of z^0_ζ is reflected in the actual solution z_ζ when $1 >> |\zeta - \zeta_s(t)| >> \mathcal{B}^{\frac{2}{3(2-\beta)}}$. Thus, in the outer asymptotic sense, the solution $z(\zeta, t)$ behaves like $z^0(\zeta, t)$ near $\zeta_s(t)$. When $|\zeta_s(t)| \to 1$, as must happen for every singularity at the late stages, the physical interface can become quite distorted locally. In this final stage, our analysis is restricted to $\frac{1}{2} \le \beta < 1$. We find that the governing equations when $|\zeta_s(t)| - 1 = O(\mathcal{B}^{1/(3(1-\beta))})$ is nonlocal in an inner scale where $\zeta - \zeta_s(t) = O(\mathcal{B}^{1/(3(1-\beta))})$. We have not attempted to solve this inner equation; however, the z_ζ in this inner neighborhood scales as $\mathcal{B}^{-\beta/(3(1-\beta))}$), which is large for small $\mathcal{B}$. We conclude that when a singularity of this kind is present in the initial data in $|\zeta| > 1$, it will eventually cause locally large localized distortions of the interface when $|\zeta_s(t)| \to 1$.

For an initial zero of $z_\zeta(\zeta, 0)$ that is independent of $\mathcal{B}$, we find that while $z^0(\zeta, t)$ would have merely moved the zero to a new location $\zeta_0(t)$, the effect of surface tension is create new singularities immediately. The possibility of these singularities, where z_ζ has a -4/3 power singularity just like the ones for an initial branch point singularity was recognized by Constantin & Kadanoff (1991) and Howison (1991). We actually confirm their existence and calculate their locations at early times. Further, unlike the case of initial branch points, where such singularity locations are within an inner region around $\zeta_s(t)$, which masks the effects of the -4/3 singularities, in this case, each of the -4/3 singularities move away from $\zeta_0(t)$. For $t = O(1)$, all these singularities are clustered over a $\mathcal{B}^{2/9}$ scale. However, the effect of each of these singularities $\zeta_p(t)$ are felt over a small $\mathcal{B}^{7/18}$ neighborhood around each of these points beyond which the solution matches to $z^0(\zeta, t)$. In this region of singularity cluster created by an initial zero $\zeta_0(0)$, there is also a point

$\zeta_{s_1}(t)$, where the perturbation expansion (1.11) breaks down because z^1 is singular and higher order terms z^2, z^3, etc. are progressively more singular. However, this is not a true singularity of z_ζ, as the solution is smoothed out in an inner scale of $O(\mathcal{B}^{1/3})$ extent where the deviation of z_ζ from z_ζ^0 is at best $O(\mathcal{B}^{1/6})$. This inner regions around each $\zeta_p(t)$ and $\zeta_{s_1}(t)$ do not overlap. Our results here become invalid when any of $|\zeta_p(t)|$ or $|\zeta_{s_1}(t)|$ is very close to 1, as in that case new scalings of the inner variables become necessary that lead to new nonlocal equations that are yet to be analyzed. We are of the opinion that Dai et al's (1991) direct numerical computation reflecting the creation of new singularities out of an initial zero is reflected in our findings, though the detected 'pole-like' singularity in the numerical calculations may be an artifact of the clustering of the singularities over a $\mathcal{B}^{2/9}$ scale. One of the immediate implications of our findings is that it is possible for z_ζ to deviate from z_ζ^0 by $O(1)$ or larger even at points where z_ζ^0 is neither zero nor singular.

Our results for small nonzero $\mathcal{B}$ suggests that we can start with smooth physical interface initally; at later times the interface can get highly convoluted over a local length scale, as the effect of singularities either present initially, or those created from an initial zero of z_ζ continually move towards $|\zeta| = 1$. Since the location of initial singularity and zeros is highly sensitive to arbitrarily small changes of $z(\zeta, 0)$ in the physical domain $|\zeta| \leq 1$, there will be highly sensitive dependence of observed features on initial conditions.

2. Analytically continued equations in $|\zeta| > 1$

Using Poisson integral formulae that relates a harmonic function and its conjugate to its boundary value on the unit circle, from (1.9) for $|\zeta| < 1$,

$$\omega = -\mathcal{B}\, I_1 \tag{2.1}$$

where

$$I_1 = \frac{1}{2\pi i} \oint_{|\zeta|=1} d\zeta' \left[\frac{\zeta + \zeta'}{\zeta' - \zeta}\right] \frac{I_2(\zeta')}{{z_\zeta}^{1/2}(\zeta', t)} \left[1 + \frac{1}{2}\zeta' \frac{z_{\zeta\zeta}(\zeta', t)}{z_\zeta(\zeta', t)} + \frac{1}{2} I_3(\zeta', t)\right] \tag{2.2}$$

where

$$I_2 = \frac{1}{\tilde{z}_\zeta^{1/2}(\frac{1}{\zeta}, t)} \tag{2.3}$$

$$I_3 = \frac{1}{\zeta} \frac{\tilde{z}_{\zeta\zeta}(\frac{1}{\zeta}, t)}{\tilde{z}_\zeta(\frac{1}{\zeta}, t)} \tag{2.4}$$

where for the channel geometry,

$$\tilde{z}(\zeta, t) = z(\zeta, t) \tag{2.5}$$

and for the radial geometry:

$$\tilde{z}(\zeta,\,t) \;=\; \frac{a(t)}{\zeta} \;+\; \sum_{n=0}^{\infty} k_n^*(t)\,\zeta^n \tag{2.6}$$

where superscript * denotes complex conjugate. By a standard method of analytic continuation through contour deformation (see Tanveer, 1987a for an example), for $|\zeta| > 1$,

$$\omega \;=\; -\mathcal{B}\,I_1 \;-\; \mathcal{B}\,I_2\,\frac{1}{{z_\zeta}^{1/2}}\left[2 \;+\; \zeta\,\frac{z_{\zeta\zeta}}{z_\zeta} \;+\; I_3\right] \tag{2.7}$$

Alternatively, (2.7) follows from (2.1) on using a variation of the well known Plemelj formula (see for example Carrier, Krook & Pearson (1983)). Note the choice of branch of half power appearing in (2.3) and (2.7) should be consistent with $|z_\zeta| \;=\; {z_\zeta}^{1/2}\,\tilde{z}_\zeta^{1/2}$ on $\zeta \;=\; e^{i\nu}$.

The analytical continuation leading to (2.7) can alternately be done by noting that on $\zeta \;=\; e^{i\nu}$, an analytic function $G(\zeta)$ in $|\zeta| \;\leq\; 1$ with representation:

$$G(\zeta) \;=\; \sum_{n=0}^{\infty} g_n\,\zeta^n \tag{2.8}$$

satisfies the relation: $2\;Re\;G \;=\; G(\zeta,t) \;+\; \tilde{G}(\frac{1}{\zeta},\,t)$, where

$$\tilde{G}(\zeta,t) \;=\; \sum_{n=0}^{\infty} g_n^*\,\zeta^n \tag{2.9}$$

Applying this to (2.9), we obtain (2.7) with

$$I_1 \;=\; \frac{1}{\mathcal{B}}\,\tilde{\omega}(\frac{1}{\zeta},\,t) \tag{2.10}$$

This provides an alternate expression for I_1 besides (2.2).

Using Poisson's integral formulae for the harmonic function $Re\;[\frac{z_t}{\zeta z_\zeta}]$, with appropriate choice of imaginary constant for $|\zeta| \;<\; 1$, (1.10) implies:

$$z_t \;=\; \zeta\,z_\zeta\,I_4 \tag{2.11}$$

where

$$I_4\,(\zeta,\,t) \;=\; \frac{1}{4\pi i}\oint_{|\zeta|=1}\frac{d\zeta'}{\zeta'}\left[\frac{\zeta \;+\; \zeta'}{\zeta' \;-\; \zeta}\right]\left[\frac{\zeta' W_\zeta(\zeta',\,t)}{z_\zeta(\zeta',\,t)}\,I_5(\zeta',\,t) \;+\; \frac{I_6(\zeta',\,t)}{z_\zeta(\zeta',\,t)}\right] \tag{2.12}$$

where

$$I_5 = \frac{1}{\tilde{z}_\zeta(\frac{1}{\zeta},\, t)} \tag{2.13}$$

$$I_6 = \frac{1}{\zeta}\, \frac{W_\zeta(\frac{1}{\zeta},\, t)}{z_\zeta(\frac{1}{\zeta},\, t)} \tag{2.14}$$

By using analytic continuation through contour deformation, for $|\zeta| > 1$ we obtain:

$$z_t = \zeta z_\zeta\, I_4 + \zeta^2\, W_\zeta\, I_5 + \zeta\, I_6 \tag{2.15}$$

Alternately, if we use the procedure of the last paragraph, we can derive (2.15) provided

$$I_4 = -\frac{\zeta\, \tilde{z}_t\, (\frac{1}{\zeta},\, t)}{\tilde{z}_\zeta\, (\frac{1}{\zeta},\, t)} \tag{2.16}$$

Equation (2.16) provides an alternate expression for I_4 (besides (2.12)) for $|\zeta\,| > 1$.

Combining (2.7) and (2.15), we obtain one nonlinear integro-differential equation for z for $|\zeta| > 1$:

$$z_t = q_1\, z_\zeta + q_2 + \mathcal{B} q_3 + \mathcal{B}\frac{q_4}{{z_\zeta}^{1/2}} + \mathcal{B}\frac{q_5 z_{\zeta\zeta}}{{z_\zeta}^{3/2}} - \frac{3}{2}\,\mathcal{B}\frac{q_7 {z_{\zeta\zeta}}^2}{{z_\zeta}^{5/2}} + \mathcal{B}\frac{q_7 z_{\zeta\zeta\zeta}}{{z_\zeta}^{3/2}} \tag{2.17}$$

where

$$q_1 = \zeta\, I_4 \tag{2.18}$$

$$q_2 = \zeta^2 I_5\, W_\zeta + \zeta\, I_6 \tag{2.19}$$

$$q_3 = -\zeta^2\, I_{1\zeta}\, I_5 \tag{2.20}$$

$$q_4 = -I_{2\zeta}\, I_5\, \zeta^2\, (2 + I_3) - I_2\, I_5\, \zeta^2\, I_{3\zeta} \tag{2.21}$$

$$q_5 = -I_{2\zeta}\, I_5\, \zeta^3 + \frac{1}{2}\, \zeta^2\, I_2\, I_3\, I_5 \tag{2.22}$$

$$q_7 = -\zeta^3\, I_2\, I_5 \tag{2.23}$$

It may be noted that each of q_1 through q_7 are analytic functions of ζ outside the unit circle since they involve $\frac{z_t}{\zeta z_\zeta}$ and the z_ζ^{-1} evaluated on or inside the unit circle where they are analytic.

3. The case of zero surface tension

It is clear from that when $\mathcal{B} = 0$, $\omega = 0$ from (2.1). From (1.3), (2.11) and (2.12), for $|\zeta| < 1$,

$$z_t = \zeta\, I_4\, z_\zeta \tag{3.1}$$

where I_4 now simplies to:

$$I_4 \;=\; -\,\frac{1}{\pi^2 i}\oint_{|\zeta'|=1}\frac{d\zeta'}{\zeta'}\left[\frac{\zeta\,+\,\zeta'}{\zeta'\,-\,\zeta}\right]\frac{1}{|z_\zeta(\zeta',t)|^2} \tag{3.2}$$

For $|\zeta|\;>\;1$, (2.17) simplifies to:

$$z_t \;=\; q_1\,z_\zeta\;+\;q_2 \tag{3.3}$$

where

$$q_1\,(\zeta,\;t)\;=\;\zeta\,I_4 \tag{3.4}$$

$$q_2\;=\;-\frac{2}{\pi}\,\frac{2\,\zeta}{\tilde{z}_\zeta\,(\frac{1}{\zeta},t)} \tag{3.5}$$

Note that even though I_4 has one analytic expression (3.2), it defines two different analytic functions inside (used in (3.1)) and outside the unit circle (used in q_1 in (3.4)).

¿From the alternate expression (2.16) for I_4 , it follows that for $|\zeta|\;>\;1$

$$q_1(\zeta,t)\;=\;-\zeta^2\frac{\tilde{z}_t\,(\frac{1}{\zeta},\;t)}{\tilde{z}_\zeta\,(\frac{1}{\zeta},t)} \tag{3.6}$$

Further, noting that $\frac{1}{z_\zeta}$ and hence $\frac{1}{\tilde{z}_\zeta}$ are analytic inside the unit circle and $z_\zeta((\zeta,t)$ and $\tilde{z}_\zeta(\zeta^*,t)$ are complex conjugates on $|\zeta|\;=\;1$, it follows from (3.5) and Cauchy's integral formula that

$$q_2\;=\;-\frac{2\zeta}{\pi^2 i}\oint_{|\zeta'|=1}\frac{d\zeta'}{\zeta'-1/\zeta}\,\frac{1}{z_\zeta^*(\zeta',t)} \tag{3.7}$$

¿From (3.2), (3.4) and (3.7) it is clear that small changes of $\frac{1}{z_\zeta}$ on $|\zeta|\;=\;1$ will induce only small changes in q_1 and q_2 for $|\zeta|\;>\;1$ and so determination of q_1 and q_2 from $\frac{1}{z_\zeta}$ on the unit circle is well posed, when $|z_\zeta|$ on $|\zeta|=1$ is bounded away from zero. Further, note that if we know $z(\zeta,t)$ a circle of radius R where $R\;>\;1$ (Note R could depend on t) but smaller than the distance of the nearest singularity of $z_\zeta((\zeta,t)$ other than that at the origin, than for any $|\zeta|<R$, including $|\zeta|=1$,

$$z(\zeta,t)\;=\;-\,\frac{2}{\pi}\,ln\;\zeta\;+\;\frac{1}{2\pi i}\oint_{|\zeta'|\,=\,R}\frac{d\zeta'}{\zeta'\,-\,\zeta}\,[z(\zeta',t)\;+\;\frac{2}{\pi}\,ln\;\zeta'] \tag{3.8}$$

for the channel geometry and for the radial geometry

$$z(\zeta,t)\;=\;\frac{a(t)}{\zeta}\;+\;\frac{1}{2\pi i}\oint_{|\zeta'|\,=\,R}\frac{d\zeta'}{\zeta'\,-\,\zeta}\,z(\zeta',t) \tag{3.9}$$

where $a(t)$ is determined from:

$$a(t) = \frac{1}{2\pi i} \oint_{|\zeta'| = R} d\zeta' \, z(\zeta', t) \tag{3.10}$$

The determination of z for $|\zeta| < R$ from given z on $|\zeta| = R$ is again well posed from the above formulae.

Further, note from (3.2) that for $|\zeta| < 1$, $Re\ I_4$ defines a harmonic function so that on $|\zeta| = 1$,

$$Re\ I_4(\zeta, t) = -\frac{1}{|z_\zeta(\zeta, t)|^2} \frac{2}{\pi} \tag{3.11}$$

Thus from the maximum principle for harmonic function for $|\zeta| < 1$:

$$\max_{|\zeta| = 1} \frac{2}{\pi} \frac{1}{|z_\zeta|^2} > -Re\ I_4 > \min_{|\zeta| = 1} \frac{2}{\pi} \frac{1}{|z_\zeta|^2} > 0 \tag{3.12}$$

Further, a little manipulation of (3.2) gives:

$$I_4(\frac{1}{\zeta_1}, t) = \frac{1}{2\pi^2 i} \oint_{|\zeta'|=1} \frac{d\zeta'}{\zeta'} \left[\frac{\zeta_1 + \zeta'}{\zeta' - \zeta_1}\right] \frac{1}{|z_\zeta(\frac{1}{\zeta'}, t)|^2} \tag{3.13}$$

It is clear that $Re\ I_4(\frac{1}{\zeta_1}, t)$ is a harmonic function in $|\zeta_1| < 1$ taking on the boundary value:

$$Re\ I_4(\frac{1}{\zeta_1}, t) = \frac{1}{|z_\zeta(\frac{1}{\zeta_1}, t)|^2} \frac{2}{\pi} \tag{3.14}$$

on $|\zeta_1| = 1$. By putting $\zeta_1 = \frac{1}{\zeta}$, from maximum principle, it follows that any $|\zeta| > 1$,

$$\max_{|\zeta| = 1} \frac{1}{|z_\zeta|^2} \frac{2}{\pi} > Re\ I_4 > \min_{|\zeta| = 1} \frac{1}{|z_\zeta|^2} \frac{2}{\pi} > 0 \tag{3.15}$$

The properties (3.12) and (3.15) will be crucial as we shall now see.

The ensuing discussion will focus on solutions to (3.3) with initial data obtained by analytically continuing physically relevant initial data $z(\zeta, 0)$, subject to the decomposition (1.1) (or 1.5 for the radial case)) and the requirement that $z_\zeta(\zeta, 0) \neq 0$ in $|\zeta| \leq 1$. There is no apriori reason for a specific solution of the analytically continued equation (3.3) for $t > 0$ to be identical to the solution to the original equations (satisfying boundary condition (1.10)) with decomposition (1.1) (or (1.5) for the radial case). Therefore, as part of our discussion, we will include analytical arguments that suggest that the solution to (3.3) that we calculate in $|\zeta| > 1$, on analytic continuation back to $|\zeta| < 1$, is consistent with decomposition (1.1) (or 1.5 for the radial case) and that with a finite speed of any zero of z_ζ, there will be no zero of z_ζ in $|\zeta| \leq 1$, atleast for a finite nonzero time interval.

Equation (3.3) is in some sense a nonlocal hyperbolic p.d.e, if indeed hyperbolicity can be defined for such equations. The coefficients are clearly analytic for $|\zeta| > 1$, except at infinity, where from (3.5) and (3.6), q_1 is proportional to ζ , but q_2 is analytic. In reality, (3.3) is a nonlinear integro-differential equation. The analyticity of the coefficients q_1 and q_2 , however, guarantee that there is no spontaneous generation of singularity in $|\zeta| > 1$ for if it were otherwise, reversal of time would imply that the singularity moves somewhere else rather than disappear (see Lacey, 1982 for a discussion of moving singularities in a differing but equivalent formulation). Thus any singularity of $z(\zeta, t)$ outside the unit circle at a point must be the result of singularities moving from somewhere else. Earlier, Richardson (1972) showed the invariance of singularity form using a Schwartz function approach. Even though, he was concerned with a displacement of a less viscous fluid by a more viscous fluid rather than the other way around, his results hold by using time reversal argument. However, the motion of singularities was not addressed in the Richardson (1972) paper.

We note from (3.3) that the characteristic velocity is $-q_1$ that is always pointed inwards towards $|\zeta| = 1$.

¿From (3.4) and (3.15), it follows that

$$Re \left[\frac{\dot{\zeta}_c(t)}{\zeta_c(t)} \right] = - Re \; I_4(\zeta_c(t), t) < 0 \tag{3.16}$$

on a characteristic $\zeta_c(t)$. Equation (3.16) immediately implies that the radial component of characteristic speed at any time t at any ζ outside the unit circle is always inwards towards $|\zeta| = 1$ and so information in $|\zeta| > 1$ flows inwards towards $|\zeta| = 1$ at all times for which a solution exists. In particular, this implies that singularities of the initial conditions in $|\zeta| > 1$ move towards the unit circle since a singularity must move with the characteristic speed $-q_1$ evaluated at that point.

On the other hand, for $|\zeta| < 1$, for given I_4 , (3.1) is a hyperbolic equation with characteristic speed $-\zeta \, I_4$, which according to (3.12), moves outwards towards $|\zeta| = 1$. Further the characteristic speed is zero at $\zeta = 0$ and so any singularity of z at $\zeta = 0$ at initial time will not move in time.

¿From the exterior equation (3.3), as $\zeta \to \infty$, using (1.1), (3.5) and (3.6) the asymptotic equations are:

$$z_t = \frac{\pi}{2} f_t(0, t) \, \zeta \, z_\zeta + 2 \tag{3.17}$$

for the channel case and for the radial geometry from (1.5), (3.5) and (3.6), we get:

$$z_t = \frac{a'(t)}{a(t)} \, \zeta \, z_\zeta + \frac{4}{\pi \, a(t) \zeta} \tag{3.18}$$

The general solution to the initial value problem in (3.17) is

$$z(\zeta,t) \;=\; 2\,t \;+\; z(e^{\frac{\pi}{2}(f(0,t)-f(0,0))}\zeta,\; 0) \tag{3.19}$$

and for (3.18), it is:

$$z(\zeta,t) \;=\; \frac{4t}{\pi\; a(t)\zeta} \;+\; z\left\{\frac{a(t)}{a(0)}\;\zeta,\; 0\right\} \tag{3.20}$$

Thus any singularity present at infinity at initial time does not move to finite ζ values.

Now consider the motion of a zero $\zeta_0(t)$ of z_ζ. Here, by a zero of z_ζ, we only refer to points where z_ζ is zero but analytic. If z_ζ is zero but nonanalytic at a point (say like a branch point), then it is called a singular point and will be denoted as $\zeta_s(t)$ just like any other singularity.

To follow the motion of zeros, it is convenient to take the derivative of (3.3) with respect to ζ and divide the resulting expression by $-z_\zeta^2$ to obtain:

$$(\frac{1}{z_\zeta})_t \;-\; q_1\,(\frac{1}{z_\zeta})_\zeta \;=\; -\frac{q_{1\zeta}}{z_\zeta} \;-\; \frac{q_{2\zeta}}{z_\zeta^2} \tag{3.21}$$

¿From the form (3.21), one notes that a double zero of z_ζ is only possible when $q_{2\zeta} \;=\; 0$. This can occur momentarily; however, we limit our discussion to a simple zero for which

$$z_\zeta \;\sim\; z_{\zeta\zeta}(\zeta_0(t),t)\;(\zeta-\zeta_0(t)) \tag{3.22}$$

On substituting this into (3.21), we find

$$\dot{\zeta}_0 \;=\; -\;q_1(\zeta_0(t),t) \;-\; \frac{q_{2\zeta}(\zeta_0(t),t)}{z_{\zeta\zeta}(\zeta_0(t),t)} \tag{3.23}$$

We are unable to show in the general case that $Re\;[\frac{\dot{\zeta}_0}{\zeta_0}\;]$ is negative, i.e. if the zero must always approach the physical domain. Also, another open question is whether the velocity of approach is always smaller than $-q_1(\zeta_0(t),\; t)$, which would have been the velocity of approach of a singularity at that point. However, this is the case for the specific case of Saffman (1959) exact zero surface tension solution (in the formulation of this paper). In that case

$$z(\zeta,\; t) \;=\; i \;+\; d(t) \;-\; \frac{2}{\pi}\;ln\;\zeta \;+\; \frac{2}{\pi}\;\;(1-\lambda)\;ln\;(1-a_1\;(t)\;\zeta^2) \tag{3.24}$$

where $1 \;>\; a_1(0) \;>\; 0$ and $0 \;<\; \lambda \;<\; 1$, and $d(t)$ and $a_1(t)$ are determined by

$$\lambda\;d \;+\; \frac{1}{\pi}\,(1 \;-\; \lambda)\;ln\;a_1 \;=\; t \;+\; K_0 \tag{3.25}$$

$$d - \frac{1}{\pi} \ln a_1 + \frac{2}{\pi} (1 - \lambda) \ln (1 - a_1^2) = K_1 \tag{3.26}$$

where K_0 and K_1 and depend on the initial conditions. In this case, for $0 < \lambda < \frac{1}{2}$, we calculate $\zeta_0 = \pm \frac{i}{\sqrt{a_1(1-2\lambda)}}$ on the imaginary axis and corresponding speed $\dot{\zeta}_0$ at all times is towards to the physical domain $|\zeta| = 1$ though the magnitude of the speed is smaller than the magnitude of $-q_1(\zeta_0(t), t)$, calculated from (3.6) which is also directed towards the physical domain $|\zeta| = 1$. For $1 > \lambda > \frac{1}{2}$, within the Saffman class of solutions, $\zeta_0 = \pm \frac{1}{\sqrt{a_1(2\lambda-1)}}$, on the real positive axis at a distance further from any of the poles at $\zeta = \zeta_s = \pm \frac{1}{\sqrt{a_1}}$. The speed of each zero is directed towards unit circle, though they are less than $|-q_1(\zeta_0(t), t)|$ in absolute value. Within this class of solutions, as $t \to \infty$, the zeros actually eventually settle down at $\zeta = \pm \frac{1}{\sqrt{2\lambda - 1}}$, at finite distances from the physical domain $|\zeta| = 1$. This is unlike other cases given by Lacey (1982), Howison (1985,1986ab), Shraiman & Bensimon (1985) where the zeros actually impinge the physical domain in finite time.

Now consider the initial value problem

$$z(\zeta, 0) = -\frac{2}{\pi} \ln \zeta + i + \sum_{n=0}^{\infty} f_n(0)\, \zeta^n \tag{3.27}$$

for the channel geometry or

$$z(\zeta, 0) = \frac{a(0)}{\zeta} + \sum_{n=0}^{\infty} k_n(0)\, \zeta^n \tag{3.28}$$

for the radial geometry. We assume that z_ζ is nonzero for $|\zeta| < R_0$ and that the convergence of the infinite series occurs within a circle of radius R_0 (> 1). When this series is not convergent, we can take its unique analytic continuation across the unit circle to be the initial condition. As long as integrals (3.2) and (3.7) exist, each of q_1 and q_2 are analytic functions outside the unit circle except at $\zeta = \infty$ where q_1 is proportional to ζ (q_2 is analytic). Thus, any singularity $\zeta_s(t)$ for z outside the unit circle has to be there at initial times and owing to the hyperbolic nature of the equation, any such singularity at $\zeta_s(t)$ must move with speed $-q_1(\zeta_s(t), t)$, which owing to the property (3.16) is always towards the unit circle. This is the basic reason for ill-posedness for the initial value problem restricted to $|\zeta| \leq 1$ since two slightly different data in $|\zeta| \leq 1$ can correspond to different initial distribution of singularities: in one case the singularity may be far off while in the other case, the singularity can be made as close as we want to the physical domain. After a short time, in the latter case, depending on the singularity nature, it can impinge the physical domain and the difference between the

interfacial shapes in the two situations in any norm that measures the interfacial slope will be $O(1)$. The singularity location and strength can be arranged so that this happens for arbitrarily short times. However, when one chooses an norm that distinguishes between initial conditions based on their values in a strip adjoining the physical domain then up to certain times, the evolution problem can be expected to be well posed.

However, the problem is well posed when $|\zeta| \geq 1$ is our domain as we now argue. We assume that there exists $T > 0$ so that the speed of a singularity (given $-q_1$) or a zero (given by 3.22)) has uniform upper bound M for the inward radial component for $0 \leq t \leq T$. This appears to be reasonable assumption since all the solutions known explicitly satisfy this criteria. Then if we choose $T_1 < T$ so that $R_0 - MT_1 > 1$, then it is clear that for $t \epsilon [0, T_1]$, $R_0 - Mt > 1$ so that z is analytic for $1 \leq |\zeta| < R_0 - M t$. For such a time, each of q_1 and q_2 is determined by first determining z and therefore z_ζ on $|\zeta| = 1$ using (3.8) (or (3.9) and (3.10) for the radial case) and then using (3.4) and (3.7). Further, the value of z on a circle of radius $R_0 - M t$ will completely determine z in any other circle with radius between 1 and $R_0 - M t$ through (3.8) (or (3.9) and (3.10) for the radial flow). This determination is well posed as argued earlier. Further, it is clear that the values of z at time t determine z at any points on the closed curve S at time t shown in Fig. 5. The curve S is chosen such that it determines z on the circle of radius $R_0 - M (t + \delta t)$ at time $t + \delta t$ by using the method of characteristic on (3.3). This determination is well posed since a characteristic method is equivalent to an ordinary differential equation, for which continuous dependence on initial values is well established.

The arguments in the last paragraph suggests the well posedness of the formulation of the exterior problem. However, one must ascertain that this exterior solution to the analytically continued equation is actually in the form (1.1) or (1.5) for $T_1 \geq t > 0$ and z_ζ has no zero in $|\zeta| \leq 1$ as must be the case for a physically acceptable solution. This follows from noting that (3.1) is actually the analytic continuation of (3.3) back into $|\zeta| < 1$ across $|\zeta| = 1$. Further, at initial time, by the very construction of $z(\zeta, 0)$, the data satisfied in the interior problem is consistent with (3.27) (or (3.28) for the radial flow case) and as we mentioned before, the characteristics in the interior problem is always pointing outwards with zero characteristic speed at the origin. This means the solution of (3.1) cannot develop singularity unless I_4 is not analytic or fails to exist for $|\zeta| < 1$. However from the exterior problem $\frac{1}{z_\zeta}$ is analytic upto time T_1 on $|\zeta| = 1$ and in particular must be integrable implying I_4 exists and defines an analytic function in $|\zeta| < 1$. Thus, in the interior evolution the only singularity is at the origin in the form given in (1.1) (or (1.5) for the radial geometry). Further, no zero of z_ζ can form in $|\zeta| \leq 1$ in the

time of interest since at initial time there is no such zero and the only way for a zero to form in this domain is for $\frac{1}{2\pi i} \oint_{|\zeta'| = 1} d\zeta' \frac{1}{z_\zeta(\zeta',t)}$ to change values discontinuously in time. This can happen only if a zero of z_ζ moves through $|\zeta| = 1$. Clearly, none of the above is the the case since from the exterior problem z_ζ and its inverse are analytic on the unit circle for $t \in [0, T_1]$. Thus the solution one calculates in the exterior is actually of the form

$$z(\zeta,t) = -\frac{2}{\pi} \ln \zeta + i + \sum_{n=0}^{\infty} f_n(t) \zeta^n \tag{3.29}$$

for the channel geometry and of the form

$$z(\zeta,t) = \frac{a(t)}{\zeta} + \sum_{n=0}^{\infty} k_n(t) \zeta^n \tag{3.30}$$

for the radial geometry for $|\zeta| \leq 1$ with $z_\zeta \neq 0$ in that domain. This guarantees that the initial value problem solved in $|\zeta| \geq 1$, when analytically continued to $|\zeta| \leq 1$, is in accordance to the requirements on $z(\zeta,t)$ in $|\zeta| \leq 1$.

As a concrete example of how one might solve an initial value problem with branch point singularities, we start with initial value

$$z(\zeta,0) = \sum_{j=1}^{N} \frac{1}{(1-\beta_j)} (\zeta - \zeta_j(0))^{1-\beta_j} E_j(\zeta,0) + G(\zeta,0) \tag{3.31}$$

where $|\zeta_j| > 1$ and β_j is generally a noninteger constant and $E_j(\zeta_j(0),0) \neq 0$. In the case, when $\beta_j = 1$, we replace the above expression by the limit of $\beta_j \to 1$ which clearly gives a logarithmic expression. Further, suppose $E_j(\zeta,0)$ is entire and $G(\zeta,0)$ is an analytic function everywhere in the finite ζ plane except at $\zeta = 0$, where the singularity is consistent with the singularity of z as given by (1.1) (or (1.5)). On substituting

$$z(\zeta,t) = \sum_{j=1}^{N} (\zeta - \zeta_j(t))^{1-\beta_j} \frac{E_j(\zeta,t)}{1-\beta_j} + G(\zeta,t) \tag{3.32}$$

into (3.1), where ζ_j satisfies

$$\dot{\zeta}_j = -q_1(\zeta_j(t),t) \tag{3.33}$$

and each $E_j(\zeta,t)$ is chosen to satisfy

$$E_{j_t} - q_1 E_{j_\zeta} = (1-\beta_j) E_j \frac{[q_1(\zeta,t) - q_1(\zeta_j(t),t)]}{(\zeta - \zeta_j(t))} \tag{3.34}$$

then, it becomes clear that

$$G_t = q_1 G_\zeta + q_2 \tag{3.35}$$

On examination of (3.34) and (3.35), it is clear that for times when $|\zeta_j(t)| > 1$, and $z_\zeta \neq 0$ in $|\zeta| \leq 1$, each of E_j and G will remain analytic in $|\zeta| > 1$ since q_1 and q_2 are remain analytic outside the unit circle.

We now outline how a rigorous proof of the analyticity would follow. Consider $G(\zeta, t)$. If $\oint_C d\zeta\ G(\zeta, t) \neq 0$ for some finite contour C completely outside $|\zeta| > 1$, the preimage C_0 (which must be a closed contour) of the contour C under the flow $-q_1(\zeta, t)$ (i.e. each point of the contour moved according to the equation $\dot{\zeta} = -q_1(\zeta, t)$) at time $t = 0$ must also be a finite contour entirely in $|\zeta| > 1$ by noting the property that $Re\ [q_1/\zeta] > 0$ and finite for the time that a zero or a singularity of z_ζ has not impinged $|\zeta| = 1$. However, $\oint_{C_0} d\zeta\ G(\zeta, 0) = 0$ from the assumed analyticity initially. On the otherhand it is clear from (3.35) and the analyticity of q_2 in $|\zeta| > 1$ that $\frac{d}{dt} \oint_C G(\zeta, t) = 0$ on a contour moving with the flow $-q_1(\zeta, t)$. This leads to a contradiction, implying that $\oint_C G(\zeta, t) d\zeta = 0$ for arbitrary closed contour C in $|\zeta| > 1$, which by Moerara's theorem implies $G(\zeta, t)$ is analytic in $|\zeta| > 1$. The proof for each E_j would be along similar lines.

Thus the form of the solution (3.32) implies invariance of the singularity form at later times. Note however, that inside the unit circle at $\zeta = \frac{1}{\zeta_j^*}$, each of G and E_j may have have singularities for $t > 0$, as the analytic continuation of q_1 and q_2 has singularities inside (from (3.5) and (3.6)). However, since $z(\zeta, t)$ as argued earlier cannot have singularity inside except at $\zeta = 0$, the singularities of E_j 's and G must cancel out at $\zeta = \frac{1}{\zeta_j^*}$ on the specific Riemann sheet corresponding to the physical domain.

Given E_j and G at any instant of time on a circle of radius R greater than unity, but smaller than the nearest zero or singularity of z_ζ, (3.32) determines z on that circle which then determines z and hence z_ζ on $|\zeta| = 1$ through (3.8) (or (3.9) and (3.10) for the radial geometry). z_ζ on $|\zeta| = 1$ determines q_1 and q_2 from (3.2), (3.4) and (3.7). Equations (3.33)-(3.35) can then be used to advance E_j and G along a characteristic.

We now discuss the fate of singularities. First, we assert that a singularity cannot stay away at a finite distance from $|\zeta| = 1$ since this would otherwise imply existence of a limit point or a limit cycle where $Re\ [q_1/\zeta] = Re\ I_4 = 0$. From maximum principle and (3.15), it follows that this is impossible. Thus every singularity must indefinitely approach $|\zeta| = 1$. We know from Howison (1985,1986ab) within the context of exact solutions that pole singularities of z_ζ never actually impinge $|\zeta| = 1$ in finite time though it approaches it exponentially in time. We now discuss in a general context if a singularity $\zeta_s(t)$, once very close to the unit circle, can actually impinge the unit circle in finite time. Consider singularity in the form:

$$z_\zeta \sim B_0(t)\ (\zeta - \zeta_s(t))^{-\beta} \tag{3.36}$$

for $\zeta \rightarrow \zeta_s(t)$. On substitution into (3.3), we can easily establish

$$B_0(t) = B_0(0)\, e^{(1-\beta) \int_0^t dt_1\, q_{1\zeta}(\zeta_j(t_1),t_1)} \tag{3.37}$$

Note that for a simple pole, i.e. for $\beta = 1$, the strength $B_0(t)$ is a constant. This is known for the previous SHSB exact solutions.

Assume that at time t_0 , the singularity is very close to the unit circle. We will now find the asymptotic equation of motion for the singularity beyond that time. Let

$$\zeta_s = R\, e^{i\nu_s} \tag{3.38}$$

Since singularity moves with the characterestics, $\dot{\zeta}_s = -q_1(\zeta_s, t)$, which from (3.2) and (3.4) imply that

$$\frac{\dot{R}}{R} + i\dot{\nu}_s = -I_4(\zeta_s(t), t) \tag{3.39}$$

We now consider the asymptotic expression for I_4 as $R \rightarrow 1^+$. ¿From (3.2), we immediately get

$$-I_4(\zeta_s(t), t) = \frac{1}{\pi^2} \int_{-\pi}^{\pi} d\nu \left[\frac{R + e^{i(\nu-\nu_s)}}{e^{i(\nu-\nu_s)} - R} \right] \frac{1}{|z_\zeta(e^{i\nu}, t)|^2} \tag{3.40}$$

Let's assume $\nu_s \neq \pm\pi$ or otherwise the range of integration on ν will be changed from 0 to 2π and the same argument as given below can be repeated. We break up the integral in (3.40)

$$\int_{-\pi}^{\pi} = \int_{-\pi}^{\nu_s - \epsilon} + \int_{\nu_s + \epsilon}^{\pi} + \int_{\nu_s - \epsilon}^{\nu_s + \epsilon} \tag{3.41}$$

where

$$\epsilon = (R-1)^{1/4} \tag{3.42}$$

It is then clear that $R - 1 << \epsilon << 1$ and the integrand in each of the 1st two integrals can be replaced by

$$\frac{1}{\pi^2} \left[-i\, cot \frac{1}{2}(\nu - \nu_s) - \frac{1}{2}(R - 1)\, cosec^2 \frac{1}{2}(\nu - \nu_s) \right] \frac{1}{|z_\zeta(e^{i\nu}, t)|^2} + o(R-1) \tag{3.43}$$

On integration by parts of the second term in (3.43), one finds that the asymptotic behavior for $\beta > \frac{1}{2}$ of the sum of the first two integrals in (3.41) is given by

$$\int_{-\pi}^{\nu_s - \epsilon} + \int_{\nu_s + \epsilon} \sim -\frac{i}{\pi^2} \left[\int_{-\pi}^{\pi} d\nu\, cot \frac{1}{2}(\nu - \nu_s) \frac{1}{|z_\zeta(e^{i\nu}, t)|^2} + o(1) \right]$$

$$- \frac{(R-1)}{\pi^2} \int_{-\pi}^{\pi} d\nu \; cot\frac{1}{2}(\nu - \nu_s) \frac{\partial}{\partial \nu} \frac{1}{|z_\zeta(e^{i\nu}, t)|^2} \; + \; o(R-1) \quad (3.44)$$

The first term above is completely imaginary and will not be relevant to our discussion of the asymptotic equation for $\dot{R}$ found by taking the real part in (3.39). For $\beta < \frac{1}{2}$, one finds from integration of (3.43) the real part of of the first two integrals in (3.41) is $o((R-1)^{2\beta})$. Now consider the contribution from the third integral. Introduce new variable $s = \frac{\nu - \nu_s}{R_1}$, we find that to the leading order

$$\int_{\nu_s - \epsilon}^{\nu_s + \epsilon} \sim \frac{2\,(R-1)^{2\beta}}{\pi^2\,|B_0(t)|^2} \int_{-\frac{\epsilon}{R-1}}^{\frac{\epsilon}{R-1}} ds \, \frac{(-1-is)}{(1+s^2)} \, \{1 + \frac{2 \; - \; 2\,cos(s[R-1])}{(R-1)^2}\} \qquad (3.45)$$

For $\beta < \frac{1}{2}$, to the leading order the above expression simplifies to

$$\sim \; - \frac{4}{\pi^2} \, \frac{(R-1)^{2\beta}}{|B_0(t)|^2} \int_0^\infty (1+s^2)^{\beta - 1} \qquad (3.46)$$

On the other hand for $\beta > \frac{1}{2}$, it is clear from (3.45), that the contribution is $O((R-1)^{2\beta})$ which is smaller than $O(R-1)$ contribution to $Re\; I_4$ in (3.44). Combining the above information with the real part of (3.39) we obtain

$$\dot{R} \; \sim \; -M_1(t)\,(R-1)^{2\beta} \qquad (3.47)$$

for $R \to 1^+$ for $\beta < \frac{1}{2}$ where

$$M_1(t) \; = \; \frac{4}{\pi^2} \, \frac{1}{|B_0(t)|^2} \int_0^\infty (1+s^2)^{\beta-1} \qquad (3.48)$$

and for $\beta > \frac{1}{2}$,

$$\dot{R} \; \sim \; - \; M_2(t)\,(R-1) \qquad (3.49)$$

where

$$M_2(t) \; = \; \frac{1}{\pi^2} \int_{-\pi}^{\pi} d\nu \; cot\frac{1}{2} \, (\nu - \nu_s) \, \frac{\partial}{\partial \nu} \, \frac{1}{|z_\zeta(e^{i\nu}, t)|^2} \qquad (3.50)$$

Note that from (3.15) it follows that $M_2(t)$ as determined above must be positive. Further from (3.48), for any $\nu(t)$, $M_1(t)$ cannot go to zero when $B_0(t)$ stays bounded. Further from (3.37), and the relation of q_1 with I_4 through (3.4), it follows that $B_0(t)$ will stay bounded when $\beta > 0$. For $\beta < 0$, it is possible to have $B_0(t)$ unbounded as $|\zeta_s| \to 1$. ¿From (3.50), it is clear that that will stay bounded for $M_2(t)$ when $\beta > \frac{1}{2}$. This is true irrespective of the dependence of ν_s on t .

On solving these ordinary differential equation, we find

$$R - 1 = \left[- (1 - 2\beta) \int_{t_0}^{t} M_1(t')dt' + (R(0) - 1)^{1-2\beta} \right]^{\frac{1}{1-2\beta}} \tag{3.51}$$

for $\beta < \frac{1}{2}$ and for $\beta > \frac{1}{2}$, we find

$$R - 1 = (R_0 - 1)\, e^{-\int_{t_0}^{t} dt_1 M_2(t_1)} \tag{3.52}$$

It is clear that for $\beta < \frac{1}{2}$, $R-1$ will be zero unless $q_{1_{0\zeta}} \; (\zeta_s(t), t) \rightarrow \infty$ as $|\zeta_s| \rightarrow 1$ such that $B_0(t)$ as determined from (3.37) $\rightarrow \infty$ in such a way that the integral of M_1 in (3.51) stops growing indefinitely as $t \rightarrow \infty$. We cannot rule out such an exceptional case when $\beta < 0$, since the behavior of $I_4(\zeta_s(t), t)$ given by (3.46) implies that $q_{1_{0\zeta}} \; (\zeta_s(t), t)$ determined from (3.4) will tend to infinity as $|\zeta_s| \rightarrow 1$ for $\beta < 0$. However, more careful arguments not considered here, may rule that out as well though we are unsure about it. Thus aside from the exceptional case pointed out over here, there will be finite time singularity for $\beta < \frac{1}{2}$.

¿From (3.52), for $\beta \geq \frac{1}{2}$ the singularity never hits the the physical domain in finite time as $M_2(t)$ in (3.50) is always finite. At the cross over value of $\beta = \frac{1}{2}$, the three integral terms contribute equally and one finds a $(R-1)$ contribution for which (3.40) holds (though with a different M), and therefore in this case as well, the singularity will not reach the physical domain.

In the case of $\beta < 0$, if

$$z_\zeta(\zeta, t) \sim E_0(t) + B_0(t) \; (\zeta - \zeta_s(t))^{-\beta} \tag{3.53}$$

as $\zeta \rightarrow \zeta_s(t)$ for nonzero $E_0(t)$, then it is easy to see that $Re \; I_4(\zeta_s(t), t) = O(1)$. From the real part of (3.39), it follows that $R-1$ in this case will be zero after a finite time. Thus for a singularity of the form (3.53) with $\beta < 0$, it will move to the boundary of the physical domain in finite time.

We now sketch a numerical algorithm to solve for $z(\zeta, t)$ outside the unit circle. We will limit our description to the channel geometry though the procedure with minor changes is readily applicable to the radial flow. We will also only describe the algorithm to calculate $z(\zeta, t)$ on a circle of radius $R(t)$, where $R(t)$ shrinks with t in some prescibed way and initially does not contain any singularity of z_ζ or a zero. We will find that we can stably calculate $z(\zeta, t)$ on the unit circle, i.e. the physical interface as a by product.

Assume N uniformly spaced out points in a circle of radius $R(t)$ at any time at which $z(\zeta, t)$ is known. We now describe how to calculate $z(\zeta, t + \delta\, t)$ on a circle of radius $R(t + \delta\, t)$, where $R(t)$ is changed in some prescibed way.

¿From a discrete fast Fourier transform, we can calculate each c_j in the relation:

$$z + \frac{2}{\pi}\, ln\, \zeta = \sum_{j=-\frac{N}{2}-1}^{\frac{N}{2}} c_j\, R^j\, e^{ij\, \nu} \tag{3.54}$$

where $\zeta = R\, e^{i\nu}$ at points on the circle of radius $R(t)$. We know from previous discussion for the appropriate initial value problem for the continuum case that $c_j = 0$ for $j < 0$. This should be approximately true for the discrete version for large N . This provides a check on numerical accuracy. On finding the c_j , we calculate z and z_ζ for $\zeta = e^{i\nu}$ on N uniformly spaced out points using

$$z + \frac{2}{\pi}\, ln\, \zeta = \sum_{j=0}^{\frac{N}{2}} c_j\, e^{i\, j\, \nu} \tag{3.55}$$

$$z_\zeta + \frac{2}{\pi}\frac{1}{\zeta} = \sum_{j=0}^{\frac{N}{2}} j\, c_j\, e^{i\, (j-1)\, \nu} \tag{3.56}$$

Note that the c_j for negative j is not used. We now can calculate q_2 at any point outside the unit circle using (3.5) as $z = \tilde{z}$ for the channel geometry and therefore

$$q_2 = -\frac{4}{\pi}\, \frac{\zeta}{\left[-\frac{2}{\pi}\, \zeta + \sum_{j=0}^{\frac{N}{2}} j\, c_j\, \zeta^{-(j-1)}\right]} \tag{3.57}$$

Further, on finding z_ζ and hence $\frac{1}{z_\zeta}$ at N uniformly spaced out points on the unit circle, we calculate a finite set of coefficients d_j and δ_j (N of them altogether) in the Fourier series representation of

$$\frac{2}{\pi |z_\zeta(e^{i\nu}, t)|^2} = d_0 + \sum_{j=1}^{\infty} d_j\, cos\, (j\, \nu + \delta_j) \tag{3.58}$$

¿From (3.2) and (3.4), it follows that

$$q_1 = -\zeta \left[d_0 + \sum_{j=1}^{\infty} d_j\, e^{i\delta_j}\, \zeta^{-j}\right] \tag{3.59}$$

After knowing d_j in (3.58), a finite truncation of (3.58) can be used to calculate q_1 at any ζ point outside the unit circle.

Now, we put (3.3) in a characteristic form and consider the problem of determining z on a circle of radius $R(t + \delta\ t)$ chosen such that

$$-ln\ R(t+\delta t)\ +\ R(t)\ =\ \max_{|\zeta|\ =\ R(t)} \left[Re\ \frac{q_1}{\zeta} \right]\ \delta t \tag{3.60}$$

Now the value of z at a point P at time $t\ +\ \delta\ t$ in Fig. 5 is determined from value of z , q_1 and q_2 at point P_1 on the curve S at time t and q_1 , q_2 at P at time $t\ +\ \delta t$. Applying a second order difference scheme for the characterestic, we obtain:

$$\zeta(P)\ -\ \zeta(P_1)\ =\ -\frac{1}{2}\ [\ q_1(\zeta(P_1),t)\ +\ q_1(\zeta(P),\ t\ +\ \delta t)\ ]\ \delta t \tag{3.61}$$

$$z(\zeta(P),t\ +\ \delta)\ =\ z(\zeta(P_1),t)\ +\ \frac{1}{2}\ [q_2(\zeta(P_1),t)\ +\ q_2(\zeta(P),t+\delta t)\ ]\ \delta t \tag{3.62}$$

In the above, $\zeta(P_1)$ and $z(P,t\ +\ \delta t)$ for N uniformly spaced out points P on the circle of radius $R(t\ +\ \delta t)$ have to be determined all at once in an iterative fashion since each of q_1 and q_2 at time $t\ +\ \delta t$ are determinable in terms of z at $t+\delta t$ on N uniformly spaced out points on circle of radius $R(t\ +\ \delta t)$ (as described in the last paragraph).

The procedure can be repeated until $R(t\ +\ \delta t)\ \leq\ 1$ or $z_\zeta\ =\ 0$ on the unit circle for which q_1 will fail to exist. We note that to follow the motion any singularity, it is not necessary to calculate z near that singularity. Once q_1 is calculated as in the above procedure, it is known for every ζ outside the unit circle and we can then calculate each singularity location by numerically integrating the ordinary differential equation $\dot{\zeta}_s\ =\ -q_1(\zeta_s(t),t)$. The strength of the singularity $B_0(t)$ given by (3.37) can similarly be calculated at any time through integration.

In the following sections, we consider how small nonzero surface tension modifies the dynamics of $z(\zeta,t)$. Our discussion of surface tension effects will be limited to initial conditions that are independent of $\mathcal{B}$. Further, we will assume that the singularity of z_ζ at initial time is either a simple pole or a branch point of a power type like any of the ζ_j in the expression (3.31) with $0 < \beta < 1$. We will also assume that the zeros and singularities of $z_{0\zeta}$ are at $O(1)$ distance from each other. Further, our results in sections 4 to 7 are restricted to cases when the distance of each singularity of z_ζ from the boundary of the physical domain, $|\zeta| = 1$, is $O(1)$. However, since the physical interface is $|\zeta| = 1$, the dynamics described in these sections are not directly relevant at that instant of time to the large distortions of interfacial shape caused by one or more singularities coming very close to $|\zeta| = 1$. For large times, all singularities must come very close to $|\zeta| = 1$ and the study of the this latter stage is taken up in section 8, though in a limited way.

4. Perturbation expansion in powers of $\mathcal{B}$.

The discussion in the last section suggest that the zero surface tension problem solved in the unphysical plane is well posed. Thus, it is natural to study the effect of small non zero surface tension in this unphysical plane. We start by assuming that an asymptotic expansion for some part of the complex ζ plane at least in the vicinity of the physical domain is of the form

$$z \sim z_0 + \mathcal{B} z_1 + ... \tag{4.1}$$

In this section, we will examine where such a perturbation expansion is inconsistent. In latter sections, we then proceed to show that under certain assumptions, a so called 'inner' region exists (in what is now the standard language in the theory of matched asymptotic expansion) such that an appropriate inner-outer matching is possible.

On substituting (4.1) into (2.17), it is clear that z_0 denotes the z determined for $\mathcal{B} = 0$ in the last section. Going back to (2.17), it is clear that z_1 satisfies the integro-differential equation:

$$z_{1t} - q_{1_0} z_{1\zeta} = q_{2_1} + q_{1_1} z_{0\zeta} + q_{3_0} + \frac{q_{4_0}}{z_{0\zeta}^{1/2}} + \frac{q_{5_0} z_{0\zeta\zeta}}{z_{0\zeta}^{3/2}} - \frac{3}{2} \frac{q_{7_0} z_{0\zeta\zeta}^2}{z_{0\zeta}^{5/2}} + \frac{q_{7_0} z_{0\zeta\zeta\zeta}}{z_{0\zeta}^{3/2}} \tag{4.2}$$

where subscript 0 on each of q_1 , q_3 , q_4 , q_5 and q_7 denote the evaluation of this quantities as given by (2.18), (2.20)-(2.23) using $\omega = 0$ and $z = z_0$ as determined in the last section. The subscript 1 on q_1 and q_2 denotes the derivatives of each of q_1 and q_2 with respect to $\mathcal{B}$ at $\mathcal{B} = 0$ as determined from (2.18) and (2.19) on substituting (4.1) for z . We can solve (4.2) in the unphysical plane by a variation of the numerical method outlined in the last section for zero surface tension since the operator on the left hand side is similar to that of the equation for z_0 (in 3.3). It is clear from (4.2) that at points where $z_{0\zeta}$ is either singular or has a zero, there is singular forcing on the right hand side of (4.2). In the case when $z_{0\zeta} \sim B_0(t) \left(\zeta - \zeta_s(t)\right)^{-\beta}$ as $\zeta \to \zeta_s(t)$, with $\beta < \frac{4}{3}$ where $B_0(t) = E_j(\zeta_s(t), t)$ for $E_j(\zeta, t)$ determined by (3.34) with the identification $\zeta_j = \zeta_s$, we find from (4.2)

$$z_{1\zeta} \sim B_1(t) \left(\zeta - \zeta_s(t)\right)^{\frac{1}{2}\beta - 3} \tag{4.3}$$

where $B_1(t)$ is determined by the first order differential equation

$$\dot{B}_1 - [\frac{1}{2} \beta - 2] q_{1_{0\zeta}}(\zeta_s(t), t) B_1(t) = \frac{1}{4} q_{7_0}(\zeta_s(t), t) B_0(t)^{-1/2} \beta (\beta + 2) (\beta + 4) \tag{4.4}$$

with initial condition:

$$B_1(0) = 0 \tag{4.5}$$

Note that $\beta = 1$, i.e. the singularity is a simple pole for z_ζ , $B_0(t)$ is a constant as noted before in the context of SHSB solutions. For $\frac{4}{3} < \beta < 2$, the most singular term on the right hand side of (4.2) comes from $q_{1_1}\, z_{0\zeta}$ rather than the curvature terms. However, the uniformity of (4.1) arising because of such singular terms is simply because the singularity of the outer expansion $z_0 + \mathcal{B}\, z_1 + ..$ travels with speed q_1 rather than q_{1_0} . Thus in approximating the location of the singularity by ζ_s , we commit an error in the location of the singularity of order $\mathcal{B}$, which shows up in the regular expansion (4.1) in term z_1 as a more singular term $(\zeta - \zeta_s)^{-\beta-1}$ than that of $z_{0\zeta}$. For β in the interval $(4/3,\ 2)$, this happens to be more singular than the curvature term which also induces in $z_{1\zeta}$ a more singular term than $z_{0\zeta}$. This nonuniformity can be remedied in the two term outer perturbation $z_0 + \mathcal{B}\, z_1$ by determining the singularity position $\zeta_s(t)$ of z_0 using

$$\dot{\zeta}_s = -q_{1_0}(\zeta_s(t), t) - \mathcal{B}\, q_{1_1}(\zeta_s, t) \tag{4.5}$$

However, even with this remedy, there are still singular terms on the right hand side of (4.2) arising from the curvature, the most singular of which induces a singularity of the form (4.3) which is more singular than $z_{0\zeta}$ when $\beta < 2$. For $\beta > 2$, there is no nonuniformity of (4.1) owing to curvature, as the induced singularity in $z_{1\zeta}$ is less singular than $z_{0\zeta}$ in that case.

The conclusion is that if the initial value problem has a singularity for $z_\zeta \sim B_0(0)\,(\zeta - \zeta_s(0))^{-\beta}$ $|\zeta_s(0)| > 1$, and if that initial condition is independent of $\mathcal{B}$, then the singularity form is preserved by $z_{0\zeta}$ though its strength $B_0(t)$ generally changes (simple pole is an exception) in time; the higher order perturbation term has a worse singularity at approximately the same point (to order $\mathcal{B}$) for $\beta < 2$. Thus, one can expect an inner region near any such $\zeta_s(t)$, where the asymptotic expansion (4.1) becomes invalid. This inner region will be treated in the next section.

Going back to (4.2), we find that the right hand side forcing terms is singular when $z_{0\zeta}$ is analytic but zero (generically simple). Thus, if we consider an initial value problem for $z(\zeta, t)$ with initial conditions that are independent of surface tension $\mathcal{B}$ such that $z_\zeta(\zeta, 0) = 0$ at some point, then we know from section 3 that $\mathcal{B} = 0$ solution $z_{0\zeta}$ preserves this zero $\zeta_0(t)$ that move in time according to (3.23). At such a point $\zeta_0(t)$, one finds a singular forcing on the right hand side of (4.2). Near $\zeta_0(t)$, $z_{0\zeta}(\zeta, t) \sim z_{0\zeta\zeta}\,(\zeta_0(t), t)\,(\zeta - \zeta_0(t))$ and this induces a singular term in z_1 at $\zeta_0(t)$ so that the leading order asymptotic behavior is

$$z_{1\zeta}(\zeta, t) \sim A_0(t)\,(\zeta - \zeta_0(t))^{-5/2} \tag{4.6}$$

as $\zeta \to \zeta_0(t)$ obtained by substituting (4.6) into (4.2) and equating the most singular

term. On using (3.23), we find that

$$A_0(t) \;=\; -\,\frac{3}{2}\,\frac{[z_{0\zeta\zeta}(\zeta_0(t),t)]^{-\frac{1}{2}}\; q_{7_0}(\zeta_0(t),t)}{q_{2\zeta}(\zeta_0(t),t)} \tag{4.7}$$

However (4.6) does not satisfy the initial condition $z_1(\zeta,0) \;=\; 0$, as it must for $\mathcal{B}$ independent initial condition $z(\zeta,0)$. Therefore, to a particular solution that behaves like (4.6), one must add in a solution to the homogeneous equation on left hand side of (4.2) so that initial condition is satisfied. The homogeneous part of the solution must have a singularity at $\zeta \;=\; \zeta_0(0)$ at $t \;=\; 0$ so as to cancel the singularity in (4.7) at $t \;=\; 0$. However, if there is a singularity present in the homogenous solution at $t \;=\; 0$, it must be move at the singularity speed $-q_{1_0}$ (from left hand side of (4.2)) which is different from the speed of a zero given by (3.23). Combining all the above information, we conclude that for $t \;>\; 0$, there must be a singular point ζ_{s1} of $z_{1\zeta}$ that initially coincides with $\zeta_0(0)$, i.e. $\zeta_{s1}(0) \;=\; \zeta_0(0)$, whose location at later times is determined by $\dot{\zeta}_{s_1} \;=\; -q_{1_0}\,(\zeta_{s1}(t),t)$ such that as $\zeta \;\rightarrow\; \zeta_{s1}(t)$,

$$z_{1\zeta} \;\sim\; A_1(t)\;\;(\zeta \;-\; \zeta_{s1}(t))^{-5/2} \tag{4.8}$$

where

$$A_1(t) \;=\; A_1(0)\; e^{-\frac{3}{2}\int_0^t\, dt'\; q_{1_{0\zeta}}(\zeta_{s_1}(t'),t')} \tag{4.9}$$

and $A_1(0) \;+\; A_0(0) \;=\; 0$. Thus one observes the birth of a new singular point $\zeta_{s1}(t)$ of the outer asymptotic expansion (4.1) that is neither a singular point of z_0 nor a zero of $z_{0\zeta}$, except at initial time. Thus, aside from each singular point of $z_{0\zeta}$ discussed earlier, an inner region is required near each zero $\zeta_0(t)$ zero of $z_{0\zeta}$ (which is a singularity point of z_1), and around each new singularity $\zeta_{s1}(t)$ of z^1. (Note: this is not a true singularity of z_ζ, as there is smoothing over an inner scale as shown in section 7.) The scale of the the inner region near $\zeta_0(t)$ is proportional to $\mathcal{B}^{2/7}$, whereas the difference of speed of the singularities between $\zeta_0(t)$ and ζ_{s1} is order unity. This means that for $t \;=\; O(\mathcal{B}^{2/7})$, the location of ζ_{s1} is within the inner scale around $\zeta_0(t)$ and the two term (in $\mathcal{B}$) outer expansion that one can expect to match with the two term asymptotic behavior of the solution to the leading order inner equation for large values of inner spatial variable is (from Van Dyke's matching principle)

$$z \;\sim\; z_{\zeta\zeta}\,(\zeta_0(t),t)\,(\zeta \;-\; \zeta_0(t)) \;+\; \mathcal{B}\,\left[\,A_0(t)\,(\zeta - \zeta_0(t))^{-5/2} \;+\;\; A_1(t)\;\;(\zeta \;-\; \zeta_{s1}(t))^{-5/2}\right] \tag{4.10}$$

The matching question will be taken up in section 6. However, if indeed a successful inner outer matching is possible, the evidence presented in this section suggests that new

singularities for $z_{1\zeta}$ are born and these will move towards the physical domain. For the case of Saffman solution, each such newly created ζ_{s_1} moves faster than a zero since we showed in section 3 that the speed of $\zeta_0(t)$ is smaller than $-q_1(\zeta_0(t), t)$ for this class of solutions. We emphasize that these singularities like the singularities and zeros of $z_{0\zeta}$ are only present in terms of the outer asymptotic solution (4.1) and that they are smoothed out in an inner scale as will be seen in section 7. In section 7, we will find that the size of each such inner region is $O(\mathcal{B}^{1/3})$ where the deviation from $z_{0\zeta}$ is $O(\mathcal{B}^{1/6})$.

Note that, if we were to look at the evolution equation for z_2 , it would contain singular forcing at the singularities of z_0 and the zeros of $z_{0\zeta}$. In addition, it will have singular forcing at the singularities of z_1 which were born initially from a zero of $z_{0\zeta}$ that moved away later. Since left hand side of the differential operator in (4.2) is the same for z_2 , z_3 , etc., as it is for z_1 , there are no new singular points for z_2 (or z_3 , etc) other than ones for z_1 .

Note that the singularity structure for z_1 and z_2 and other higher order terms that we have sketched out are dependent on the initial condition being independent of $\mathcal{B}$. If we have initial conditions that allow appropriate terms in z_1 , one may not have any birth of a singularity of z_1 from an initial zero of z_ζ . This effect will show up at z_2 or higher order depending precisely on the initial conditions.

Aside from each point $\zeta_s(t)$, $\zeta_0(t)$ and corresponding $\zeta_{s_1}(t)$, where the perturbation expansion (1.11) fails because the higher order terms become progressively more singular, (1.11) is also invalid at any singular point $\zeta_p(t)$ created by surface tension effects (as will be seen in sections 5 and 6) on an branch point or zero of $z_{0\zeta}$, where

$$z_\zeta \sim \tilde{A}_0(t)\,(\zeta - \zeta_p(t))^{-4/3} + .. \tag{4.11}$$

as $\zeta \to \zeta_p(t)$ provided where $\tilde{A}_0(0) = 0$ and $\tilde{A}_0(t)$ scales with some some positive power of $\mathcal{B}$. Indeed, for any $\mathcal{B}$, which need not even be small, substitution of (4.11) into the complete equation (2.17) (actually its ζ derivative) shows that such a singularity is consistent with the equation with

$$\dot{\zeta}_p = -q_1(\zeta_p(t), t) - \frac{4}{9}\mathcal{B} q_7(\zeta_p(t), t)\tilde{A}_0^{-3/2} \tag{4.12}$$

Because of the scaling of $\tilde{A}_0$ with some positive power of $\mathcal{B}$, the singular behavior (4.11) is not present in $z_{0\zeta}$; further since $\tilde{A}_0(0) = 0$, there is no such singularity at initial time. However, at later times, for $t << \mathcal{B}^{-1}$, the location of singular point $\zeta_p(t)$ that initially coincides with some $\zeta_s(0)$ (where $z_\zeta(\zeta, 0)$ is singular as in (3.31) with $0 < \beta < 2$) is within an inner neighborhood of $\zeta_s(t)$ where the perturbation expansion (1.11) is invalid in any case because of the nonuniformity pointed out earlier. Thus there is no need for a separate

inner region around such $\zeta_p(t)$. However, in section (6), we shall see that a $\zeta_p(t)$ which initially coincides with with some zero $\zeta_0(0)$ moves away from $\zeta_0(t)$, but for $t = O(1)$ remains within a $O(\mathcal{B}^{2/9}$ neighborhood of $\zeta_{s_1}(t)$. However, at this time, the the inner region around $\zeta_{s_1}(t)$ which scales lik $O(\mathcal{B}^{1/3})$ does not include any of $\zeta_p(t)$. A separate inner region scaling as $\mathcal{B}^{7/18}$ around each such $\zeta_p(t)$ becomes necessary to analyze.

5. Local Nonlinear Equations near singular point of $z_{0\zeta}$

Here we consider the equation near a point where $z_{0\zeta}$ is singular generally like $(\zeta - \zeta_s(t))^{-\beta}$ for $0 < \beta < 2$.

For $\beta \geq 2$, the curvature term in the evolution equation is less singular than than other terms of the equation and there is no need for an inner zone. For $\beta < 0$, the inner equations becomes a little more complicated and is not considered here. The equations derived in this section will also be restricted by the assumption that $|\zeta_s(t)| - 1$ is not very small. Each $\zeta_s(t)$ eventually must approach $|\zeta| = 1$ as shown in section 3 and so our assumption implies that we will not be looking at the final stages of evolution when $|\zeta_s| - 1 << 1$. In section 8, however, we derive equations for the final stages of evolution for the special case $\frac{1}{2} < \beta < 1$.

Before introducing inner variables, it is convenient to take the derivative of (2.17) with respect to ζ, to come up with an evolution equation for z_ζ. Inorder that the curvature term comes at the same order as z_{ζ_t} and that the solution matches with the outer solution for which $z_{0\zeta} \sim B_0(t)(\zeta - \zeta_s(t))^{-\beta}$, it becomes necessary* to introduce inner variables:

$$\zeta - \zeta_s(t) = \mathcal{B}^{\frac{2}{3(2-\beta)}} \frac{1}{C_0(t)} \chi \tag{5.1}$$

$$z_\zeta = \mathcal{B}^{-\frac{2\beta}{3(2-\beta)}} C_0(t)\, G(\chi, \tau) \tag{5.2}$$

$$\tau = \int_0^t dt\ C_0^{3/2}(t)\, q_{7_0}\, (\zeta_s\,(t),\ t) \tag{5.3}$$

where

$$C_0(t) = C_0(0)\, e^{\int_0^t dt'\ q_{1_{0\zeta}}(\zeta_s\,(t'),t')} \tag{5.4}$$

where $C_0(0)$ depends on the initial condition $B_0(0)$. Then the leading order equation becomes:

$$G_\tau = -2\, \frac{\partial^3}{\partial \chi^3}\, G^{-1/2} \tag{5.5}$$

In deriving the above, we assumed $|\zeta_s(t)| - 1 >> \mathcal{B}^{\frac{2}{3(2-\beta)}}$, or otherwise the simplification of each of q_1 through q_7 by $q_{1_0}, .. q_{7_0}$ would be incorrect. This would make the derivation

* The term $C_0(t)$ appearing in the scaling is only meant to put the inner equation in the simplest possible form

of (5.5) invalid. When the singularity is so close to $|\zeta| = 1$, different scalings are necessary for the dependent and independent variables and instead of a local partial differential equation (5.5) for the inner equation, one obtains a parameter free integro-differential equation. We consider that in section 8 for the special case $\frac{1}{2} \leq \beta < 1$. Equation (5.5) is the Harry-Dym equation as pointed out by Kadanoff (private communiation) which can be related to the modified KdV equation (Kawamoto, 1985) and is completely integrable. However, all the inverse scattering theory results are based on the real domain where certain decay conditions hold at $\pm\infty$. These do not appear to carry over to the complex plane and we could find no direct use of the integrability of (5.5). However, there exists similarity solution that satisfy both initial and matching conditions at infinity for certain ranges of arguments of $Arg\ \xi$.

Consider the initial value problem, with

$$G(\chi, 0) = \frac{1}{\chi^{\beta}} \tag{5.6}$$

with boundary conditions

$$G(\chi, \tau) \sim \frac{1}{\chi^{\beta}} \tag{5.7}$$

for χ in some sector of the complex χ plane that will be determined shortly. Equation (5.7) corresponds to assuming that at initial time, z_{ζ} is the same as $z_{0\zeta}$. The boundary condition (5.7) corresponds to the leading order inner-outer matching, i.e. as we move out of the inner zone the solution is assumed to match to $z_{0\zeta}$.

It is convenient to transform the Harry Dym equation (5.5)

$$H = G^{-1/2} \tag{5.8}$$

Then the equation for H is:

$$H_{\tau} = H^3\, H_{\chi\chi\chi} \tag{5.9}$$

This has a similarity solution that satisfies both initial and boundary conditions (5.6) and (5.7) of the form

$$H = \chi^{\beta/2}\, F\left(\frac{\tau}{(\chi^{3-3\beta/2})}\right) \tag{5.10}$$

On substituting this into (5.9), we obtain

$$\frac{1}{8}\beta(\beta-2)(\beta-4)\, F + \frac{3}{8}(\beta-2)(21\beta^2-84\beta+80)\eta\, F' + \frac{27}{2}(\beta-2)^3\eta^2\, F'' + \frac{27}{8}(\beta-2)^3\,\eta^3\, F''' = \frac{F'}{F^3} \tag{5.11}$$

where

$$\eta = \frac{\tau}{\chi^{3-3\beta/2}} \tag{5.12}$$

Note that in the special case of $\beta = 1$, (5.11) can be integrated once to obtain

$$\frac{3}{8}\eta F - \frac{27}{8}\eta^2 F' - \frac{27}{8}\eta^3 F'' = \frac{1}{2}[1 - \frac{1}{F^2}] \tag{5.13}$$

where we used the boundary condition $F \rightarrow 1$ as $\eta \rightarrow 0$.

We now return to the general case of arbitrary β in the interval (0, 2). We now seek the asymptotic solution to all powers of η in order to use Borrel summation to find the leading order transcendental terms, that are beyond all orders of the asymptotic expansion. This will give information on what sectors of the complex η plane can the boundary condition $F(\eta) \sim 1$ be satisfied for small η corresponding to boundary condition (5.7) for some corresponding sector in the χ plane. It is convenient to substitute

$$F = 1 + Q \tag{5.14}$$

into (5.11) to obtain

$$-\sum_{n=1}^{\infty} (-1)^n \frac{1}{2}(n+1)(n+2) Q^n Q' + \frac{1}{8}\beta(\beta-2)(\beta-4) Q + \frac{3}{8}(\beta-2)(21\beta^2-84\beta+80)\eta Q'$$

$$+\frac{27}{2}(\beta-2)^3\eta^2 Q'' + \frac{27}{8}(\beta-2)^3 \eta^3 Q''' = Q' \tag{5.15}$$

It is clear that the complete asymptotic expression for small η is of the form:

$$Q \sim \sum_{m=1}^{\infty} a_m \eta^m \tag{5.16}$$

On substituting this into (5.15), we obtain a complicated recurrence relation. However, in order to find the leading order transcendental terms for small η, we only consider the form of the recurrence relation for large m. We will establish aposteori that the nonlinear terms will not contribute to the leading order in the ratio. Consider any nonlinear term in the summation formulae in (5.15), say QQ'. The coefficient of η^{m-1} will involve $\sum_{j=1}^{m-2} \frac{1}{2} m\, a_j\, a_{m-j}$. From the final asymptotic expression for a_m in (5.19) that is obtained by balancing the linear terms, it is not difficult to see that the the contribution from the previous summation in equation (5.15) is relatively small for large m. Similarly, we can argue for other nonlinear terms. More detailed arguments have been made by Combescot et al (1988) along these lines for a nonlinear ordinary differential equation arising in steady state selection. While, this does not rule out other forms of balance where nonlinear terms can be important, we expect that the balance obtained through linear equations is appropriate since for large $|\eta|$, transcendental terms are small in certain

sectors of the complex plane and should therefore appear as solution to a homogeneous part of equation obtained by linearizing about $F = 1$. Kruskal & Segur (1986) demonstrated this to be the case for a model nonlinear equation. ¿From ignoring the nonlinear terms in (5.15), we obtain the recurrence relation:

$$\frac{a_m}{a_{m-1}} = \frac{27}{8} (\beta - 2)^3 (m-1)^2 [1 + O(\frac{1}{m^2})] \tag{5.17}$$

This implies that

$$a_m \sim \tilde{b} \left(\frac{27}{8}(\beta - 2)^3\right)^m [(m-1)!]^2 \tag{5.18}$$

where b depends on the initial few iterates where nonlinearity in (5.15) plays a role. On using Stirling's approximation, (5.17) simplifies to:

$$a_m \sim b \left(\frac{27}{32}(\beta - 2)^3\right)^m \frac{(2m)!}{m^{3/2}} \tag{5.19}$$

where $b = \sqrt{\pi}\, \tilde{b}$. We now carry out the procedure of Borrel resummation, as previously done for the steady state fingering problem by Combescot et al. Define $R(s)$ by

$$R(s) = \sum_{m=1}^{\infty} \frac{a_m}{(2m)!} s^m \tag{5.20}$$

Then the asymptotic behavior (5.19) implies that the nearest singularity of $R(s)$ to the origin in the complex s plane is at $s = -s_0$, where $s_0 = \frac{32}{27\,(2-\beta)^3}$ and the leading order asymptotic behavior near $s = -s_0$ is given by

$$R(s) \sim b\; s_0^{-1/2}\, \Gamma(-1/2)\, (s + s_0)^{1/2} \tag{5.21}$$

The functions $R(s)$ and $F(\eta)$ are related through the transform:

$$Q(\eta) = \int_0^{\infty} ds\; e^{-s}\, R(s^2\, \eta) \tag{5.22}$$

Alternately, we can write

$$Q(\eta) = \frac{1}{\sqrt{\eta}} \int_0^{\infty} ds_1\; e^{-\frac{s_1}{\sqrt{\eta}}}\, R(s_1^2) \tag{5.23}$$

Using Watson's Lemma, it clearly follows that the asymptotic expansion of Q is as given by (5.16) for η real positive and small. This is also true for $Arg\ \eta$ in $(-\pi, \pi)$ since the contour along the real axis is not affected by the presence of singularity of $R(s)$. Notice that instead of (5.22) or (5.23), which are equivalent, we could have equally used

$$Q(\eta) = \frac{1}{\sqrt{\eta}} \int_0^{c} ds_1\; e^{-\frac{s_1}{\sqrt{\eta}}}\, R(s_1^2) \tag{5.24}$$

for any real and positive constant c that is independent of η and obtain the same asymptotic expansion (5.16) for $Arg\ \eta$ in $(-\pi,\ \pi)$. It is clear that (5.23) and (5.24) differ by transcendentally small terms in η that are beyond all orders of the asymptotic expansion (5.16). However, we shall see momentarily that (5.24) does not give the asymptotic expansion (5.16) when $Arg\ \eta$ is extended beyond π which is necessary for obtaining matching to the outer asymptotic expansion for a range of β values.

Returning to (5.22), as we continuously change argument of η from 0 to π , the stationary phase path encounters the singularity at $s = s_c = i\sqrt{\frac{s_0}{\eta}}$ on the real positive axis when $Arg\ \eta = \pi$, In conformity of analytic continuation from smaller values of $Arg\ \eta$, it follows that the path of integration should be deformed such that it goes below s_c on the real axis. For $2\ \pi > Arg\ \eta > \pi$, this deformation is equivalent to Fig. 6, which is the required stationary phase contour. The contribution from the trip around $\eta = s_c$ gives an additional contribution of

$$e^{i\ \pi/4}\ 2^{3/2}\ b\ \pi\ \eta^{1/4}\ s_0^{-1/4}\ e^{-i\sqrt{\frac{s_0}{\eta}}} \tag{5.25}$$

However, this is transcendentally small for $Arg\ \eta$ in the interval $(\pi,\ 2\ \pi)$ and hence the asymptotic expansion (5.16) holds for this range of $Arg\ \eta$. Once again, if we repeat the analysis for $Arg\ \eta$ continuously changing from 0 to $-\pi$ and going beyond $-\pi$, we find that there is an additional contribution from $s = -s_c$ owing to the use of a contour shown in figure 7 that is

$$-\ e^{-i\ \pi/4}\ 2^{3/2}\ b\ \pi\ \eta^{1/4}\ s_0^{-1/4}\ e^{i\sqrt{\frac{s_0}{\eta}}} \tag{5.26}$$

However, this is transcendentally small for $Arg\ \eta$ in the interval $(-2\pi,\ -\pi)$, Thus over $Arg\ \eta$ in the range $(-2\pi,\ 2\pi)$, the asymptotic series for Q in (5.16) hold for small η implying a matching is possible in that sector of the complex η plane to the appropriate zero surface tension solution as $\zeta \rightarrow \zeta_s(t)$. However for $Arg\ \eta$ in $(2\ \pi,\ 3\ \pi)$ and $(-3\ \pi,\ -2\ \pi)$, (5.16) does not hold as (5.25) or (5.26) become transcendentally large. At this stage, we can see the difference between (5.24) and (5.22) (or (5.23)). If we were to use (5.24), the asymptotics of Q when $Arg\ \eta$ is in $(\pi,\ 2\ \pi)$ or $(-2\ \pi, -\pi)$ will be dominated by the contribution from c which would then dominate (5.16). If c were complex with $Re\ c > 0$, it would still not be possible to ensure the validity of the asymptotic series (5.16) beyond an interval of $2\ \pi$ for $Arg\ \eta$. We now note that there is nothing special about the interval $(-2\pi,\ 2\pi)$ for $Arg\ \eta$ where the asymptotic series (5.16) is valid. Through appropriate rotation of contours in the $s-$ plane in the expression (5.22), (5.16) can be made valid for any interval of the form $(-2\pi + 2\ k\ \pi,\ 2\ k\ \pi\ +\ 2\ \pi)$ for any integer k.

Now, we examine the implications of being able to carry out an inner-outer match for a range of $Arg\ \eta$ not exceeding 4π. Clearly, the matching fails because the transcendental term (5.25) or (5.26) becomes large. These transcendental behavior, when rewritten in terms of outer variable correspond to transcendental corrections to the regular perturbation expansion (1.11) of the form

$$e^{\pm i \frac{\sqrt{s_0}[C_0(t)(\zeta-\zeta_s(t))]^{3/2-3\beta/4}}{\tau^{1/2}\mathcal{B}^{1/2}}} \tag{5.27}$$

where in (5.27), we have suppressed a prefactor that is algebraic in $\mathcal{B}$ and $\zeta - \zeta_s(t)$. This form of transcendental correction to (1.11) is only valid when $\mathcal{B}^{\frac{2}{3(2-\beta)}} << |\zeta - \zeta_s(t)| << 1$. To find the form of the transcendental terms for arbitrary ζ, one needs to look at the associated homogenous equation found by linearizing (2.17) about z_0 and look for WKB type solution to this equation. Each WKB solution obtained this way are expected to have the feature that they exchange dominancy across lines in the complex ζ plane called Stokes lines. The structure of global Stokes lines fo the steady finger problem has been analyzed before (Tanveer (1987a)). However, in the time dependent problem, the equations appear to be quite complicated and we do not address this problem here.

We now present just heuristic arguments to conclude that appropriate matching can be accomplished atleast for some range of β. These have to be checked in the future by complete determination of global Stokes lines at each instant of time. ¿From prior experience with steady problems, it appears likely, that in $|\zeta| = 1$, the dominant contribution towards the transcendental correction is from the nearest singularity. It is clear from (5.27) that the local structure of the Stokes lines near $\zeta = \zeta_s(t)$ is given by setting the real part of the exponent in (5.27) to zero. Now, from the relation (5.1), (5.3) and (5.12), it follows that

$$Arg\ \frac{(\zeta_s(t) - \zeta)}{\zeta_s(t)} = \pi - \frac{Arg\ \zeta_s(t)}{(3 - 3\beta/2)} - Arg\ C_0(t) - \frac{1}{3 - 3\beta/2}\ Arg\ \eta \tag{5.28}$$

Now, in the special case of a $\beta = 1$, i.e. a simple pole for $z_{0\zeta}$, it is clear that regardless of the value of $\pi - \frac{Arg\ \zeta_s(t)}{(3-3\beta/2)} - Arg\ C_0(t)$, an integer k can be chosen so that $Arg\ \eta$ is in $(-2\pi + 2k\pi,\ 2\pi + 2k\pi)$ when $Arg\ \frac{(\zeta_s(t)-\zeta)}{\zeta_s(t)}$ is in $(-\pi/2,\ \pi/2)$ showing that a matching is possible to the outer solution in all directions from $\zeta_s(t)$ towards the half plane that contains the physical domain (see Fig. 8). We now argue that such a matching imposed for the nearest singularity to the physical domain $|\zeta| \leq 1$, assumed to be at $O(1)$ distance, is sufficient to ensure the validity of (1.11) in the physical domain. Consider a circle around the origin in the ζ plane so that its radius R is slightly less (independent of $\mathcal{B}$) than the distance of the nearest singularity S from the origin (See Fig. 8). We know from

our discussions in the context of zero surface tension solution, that the value of z on this large circle uniquely determines z on $|\zeta| = 1$ in a well posed manner. Now if any part of this large circle contained exponentially large terms in $\mathcal{B}$ that can be expected if an inner-outer matching failed, then one or more of the coefficients of a power series representation for $z + \frac{2}{\pi} ln\ \zeta$ for the channel case ($z - \frac{a(t)}{\zeta}$ for the radial geometry) that is convergent up to this circle of radius R must contain transcendentally large terms in $\mathcal{B}$. Now since the distance of the singularity from the unit circle is assumed to be order unity (i.e. independent of $\mathcal{B}$), it follows that z on $|\zeta| = 1$ will contain transcendentally large terms in $\mathcal{B}$ that is incompatible with the assumption that (4.1) actually holds in the physical domain.

The argument just presented above suggests that for a simple pole of $z_{0\zeta}$ or β values close to unity, we are assured of a successful inner-outer matching. However, this condition may just be sufficient, not necessary. Certainly, the argument above will not hold for any $\zeta_s(t)$ which is not the nearest singularity from the physical domain. Further, the global structure of the Stokes line is likely to change due to effect of other singular points that move out of the inner region, if these are located close to a Stokes line emanating from $\zeta_s(t)$. This point is not addressed in this paper, but needs to be resolved in the future to ensure that the inner-outer structure of the solution is indeed valid for any β in the interval $(0, 2)$. Our argument only assures that for $\beta = 1$ or sufficiently close to it.

Now consider possible singularities of the actual z_ζ. It is clear that the function F appearing in the similarity solution (5.10) that satisfies (5.11) admit singularities at one or more η_0 such that

$$F(\eta) \sim \tilde{A}_1 \ (\eta - \eta_0)^{2/3} \tag{5.29}$$

as $\eta \rightarrow \eta_0$. Rewritten in terms of the outer variable, this implies that

$$z_\zeta \sim \mathcal{B}^{\frac{8/3-2\beta}{3(2-\beta)}} C_0^{-1/3}(t)(-\eta_0)^{-4/3} \left(\frac{\eta_0}{\tau}\right)^{\frac{2}{3(2-\beta)}} \tilde{A}_1^{-2} \ (3 - 3\beta/2)^{-4/3} \ [\zeta - \zeta_p(t)]^{-4/3} \tag{5.30}$$

for

$$|\zeta - \zeta_p(t)| \ << \ |[C_0(t)]^{-1} \left[\frac{\mathcal{B}\tau}{\eta_\prime}\right]^{2/(3(2-\beta))} | \tag{5.31}$$

where

$$\zeta_p(t) = \zeta_s(t) - [C_0(t)]^{-1} \left[\frac{\mathcal{B}\tau}{\eta_\prime}\right]^{2/(3(2-\beta))} \tag{5.32}$$

We now present a likely explanation to Dai et al's (1991) numerical computation that appeared to suggest that an initial pole of $z_\zeta(\zeta, 0)$ remains a pole at later times even for nonzero $\mathcal{B}$ that is small. In order to determine the nature of the singularity $\zeta_p(t)$ (5.30) on $|\zeta| = 1$ through monitoring of the power series coefficient k_n in (1.6) as carried out

by Dai et al (1991), one must consider the asymptotic behavior of k_n for very large n because of the highly localized nature of the singularity as reflected in (5.31). Further, the calculations must be accurate enough to distinguish the the small $O(\mathcal{B}^{2/9})$ (for $\beta = 1$) coefficient of the singular term in (5.30) from the behavior

$$z_\zeta \sim \frac{B_0}{\zeta - \zeta_s(t)} \tag{5.33}$$

that is valid for $\mathcal{B}^{2/3} << |\zeta - \zeta_s(t)| << 1$ given that $\zeta_s(t)$ and $\zeta_p(t)$ are quite close to each other (within $\mathcal{B}^{2/3}$ as is seen in (5.32)). We believe that the behavior (5.30) will be reflected in in k_n for very large n, far beyond what has been calculated by Dai et al (1991). In some sense, in the Dai et al (1991) calculation, the singular behavior (5.33) masks the actual singularity because sufficiently small spatial scales were not resolved.

However, due to the stiffness of the system of ODE for small $\mathcal{B}$ and larger size of truncation in the Dai et al (1991) formulation, such a direct computation appears to be impractical at this stage. This The number of k_n numerically computed by Dai et al's (1991) does not appear to be large enough and accurate enough to detect the singular behavior (5.30). Instead, their calculations only reflected the behavior: that is valid

6. Local Nonlinear Equations near a zero of $z_{0\zeta}$

As pointed out earlier, the perturbation expansion (4.1) is invalid near the zero $\zeta_0(t)$ of $z_{0\zeta}$ as well. We now present the analysis for an inner-region around $\zeta_0(t)$ which scales as $\mathcal{B}^{2/7}$, where (4.1) breaks down.

We introduce innter variables

$$\zeta - \zeta_0(t) = \mathcal{B}^{2/7} k_1 \xi \tag{6.1}$$

$$\tau = \mathcal{B}^{-2/7} \int_0^t dt' \frac{k_1(t')\, q_{2_{0\zeta}}(\zeta_0(t'), t')}{k_2(t')} \tag{6.2}$$

$$z = \mathcal{B}^{4/7} k_2 \, G(\xi, \tau) + \int^t dt' \, q_2(\zeta_0(t'), t') \tag{6.3}$$

where

$$k_1 = \frac{q_{7_0}^{2/7}(\zeta_0(t), t)}{q_{2_{0\zeta}}^{2/7}(\zeta_0(t), t)\; z_{0_{\zeta\zeta}}^{1/7}(\zeta_0(t), t)} \tag{6.4}$$

$$k_2 = \frac{q_{7_0}^{4/7}(\zeta_0(t), t)\; z_{0_{\zeta\zeta}}^{5/7}(\zeta_0(t), t)}{q_{2_{0\zeta}}^{4/7}(\zeta_0(t), t)} \tag{6.5}$$

Then equation (2.15) to the leading order in $\mathcal{B}^{2/7}$ reduces to

$$G_\tau = -G_\xi + \xi - 2\left[G_\xi^{-1/2}\right]_{\xi\xi} \tag{6.6}$$

In order that this matches with $z_{0\zeta}$ as $\zeta \rightarrow \zeta_0(t)$, we must require that

$$G_\xi(\xi, \tau) \sim \xi \tag{6.7}$$

as $\xi \rightarrow \infty$ along sectors in the complex ξ plane for any fixed scaled time τ in directions towards the physical domain. The initial conditions imply that

$$G_\xi(\xi, 0) = \xi \tag{6.8}$$

For convenience, we define

$$H = G_\xi^{-1/2} \tag{6.9}$$

Then on taking the derivative of (6.6) with respect to ξ , we find that the evolution equation for H is given by

$$-2\,\frac{H_\tau}{H^3} - 2\,\frac{H_\xi}{H^3} = 1 - 2H_{\xi\xi\xi} \tag{6.10}$$

The boundary and initial conditions (6.7) and (6.8) translate to:

$$H(\xi, \tau) \sim \xi^{-1/2} \tag{6.11}$$

for $|\xi| >> 1$ and initially

$$H(\xi, 0) = \xi^{-1/2} \tag{6.12}$$

6a. Early times $t << \mathcal{B}^{2/7}$

When $t << \mathcal{B}^{2/7}$, it is clear from the definition of τ in (6.2) that $\tau << 1$. In this case there is an asymptotic similarity solution that is relevant to boundary and initial conditions and is given by

$$H(\xi, \tau) \sim (\xi - \tau)^{-1/2}\, F\left(\frac{\tau}{(\xi - \tau)^{9/2}}\right) \tag{6.13}$$

where $F(\eta)$ ($\eta = \frac{\tau}{(\xi-\tau)^{9/2}}$ in this case) satisfies (5.11) with $\beta = -1$ and can be made to satisfy boundary condition

$$\lim_{\eta \rightarrow 0} F(\eta) = 1 \tag{6.14}$$

for $Arg\ \eta$ in $(2k\,\pi - 2\pi,\ 2k\pi + 2\,\pi)$ for any specific choice of integer k, as shown in section 5. Using the relations (6.1), (6.2) and the relation of η to τ and $\xi - \tau$, it is clear that this implies that the inner-outer matching condition appearing as boundary condition (6.11) will be valid for a $8\pi/9$ range of $Arg\ \xi$ and hence of $Arg\ (1 - \zeta/\zeta_0(t))$.

We now determine the singularity locations of $F(\eta)$ for a solution that satisfies (6.14) for $Arg\ \eta$ in the interval $(-2\pi,\ 2\pi)$. Through appropriate rotational invariances of the equation (6.6) and initial conditions, the form of the singularities for solutions that satisfy (6.14) for $Arg\ \eta$ in $(-2\pi + 2k\pi,\ 2\pi + 2k\pi)$ for nonzero integer k can then be deduced. The proper choice of integer k depends on $Arg\ k_1$ and Arg k_2 as determined in (6.4) and (6.5) as they relate the $Arg\ (1 - \zeta/\zeta_0(t))$ to $Arg\ \xi$ and hence to $Arg\ \eta$.

First, for computational purpose, we found it convenient to introduce the transformation

$$\tilde{\eta} = \eta^{-2/9} \tag{6.15}$$

with principal choice of $Arg\ \eta$ and

$$\tilde{F}(\tilde{\eta}) = \tilde{\eta}^{-1/2}\ F(\tilde{\eta}^{-9/2}) \tag{6.16}$$

Then the differential equation satisfied by $\tilde{F}$ is

$$-\frac{1}{9\tilde{F}^3}\left[\tilde{F} + 2\,\tilde{\eta}\,\tilde{F}'\right] = \tilde{F}''' \tag{6.17}$$

The boundary condition (6.14), imply that

$$\tilde{F}(\tilde{\eta}) \sim \tilde{\eta}^{-1/2} \tag{6.18}$$

Through consistent dominant balance argument on (6.17), one obtains the following possible asymptotic series at ∞:

$$\tilde{F} \sim \tilde{\eta}^{-1/2} - \frac{15}{8}\tilde{\eta}^{-5} + \frac{225 \times 115}{128}\,\tilde{\eta}^{-19/2} + \ldots \tag{6.19}$$

The form of the transcendental correction has already been determined in section (5) for $F(\eta)$ as $\eta \rightarrow 0$. Using that result, we can assure ourself that there is not transcendental term (atleast to the leading order) when $Arg\ \tilde{\eta}$ is in the interval $(-2\pi/9, 2\pi/9)$. At the anti-Stokes lines $2\pi/9$ and $-2\pi/9$, transcendental terms are born so that for $Arg\ \eta$ in the interval $(2\pi/9, 6\pi/9)$,

$$\tilde{F} \sim \tilde{\eta}^{-1/2} - \frac{15}{8}\tilde{\eta}^{-5} + \frac{225 \times 115}{128}\,\tilde{\eta}^{-19/2} + \ldots + C_1\tilde{\eta}^{-13/8}e^{i\sqrt{32/729}\tilde{\eta}^{9/4}} \tag{6.20}$$

where

$$C_1 = -e^{-i\pi/4}2^{3/2}b\pi\ s_0^{-1/4} \tag{6.21}$$

and for $Arg\ \tilde{\eta}$ in $(-6\pi/9,\ -2\pi/9)$

$$\tilde{F} \sim \tilde{\eta}^{-1/2} - \frac{15}{8}\tilde{\eta}^{-5} + \frac{225 \times 115}{128}\,\tilde{\eta}^{-19/2} + \ldots + C_2\tilde{\eta}^{-13/8}e^{-i\sqrt{32/729}\tilde{\eta}^{9/4}} \tag{6.22}$$

where

$$C_2 = e^{i\pi/4} 2^{3/2} b\pi \; s_0^{-1/4} \tag{6.23}$$

Notice that the transcendental terms in (6.20) and (6.22) are subdominant to the algebraic terms of the asymptotic expansion for $Arg\ \tilde{\eta}$ in $(2\pi/9,\ 4\pi/9)$ and $(-4\pi/9,\ -2\ \pi/9)$ respectively, but become the dominant contribution for $Arg\ \tilde{\eta}$ when the Stokes line $Arg\ \tilde{\eta}\ =\ 4\pi/9$ or $Arg\ \tilde{\eta} = -4\pi/9$ is crossed. Thus (6.19) holds for $Arg\ \tilde{\eta}$ in $(-4\pi/9,\ 4\pi/9)$ since the transcendental terms in (6.20) and (6.22) are subdominant.

We now describe a numerical procedure to calculate such a solution $\tilde{F}$ to (6.17) in the complex $\tilde{\eta}$ plane and to find the location of singularities $\tilde{\eta}_0$, where $\tilde{F}$ is singular. Earlier, Constantin & Kadanoff (1991) and Howison (1991) realized that a two-thirds singularity of $\tilde{F}$ was consistent with (6.17). Indeed, an expansion

$$\tilde{F}\ \sim\ [\tilde{\eta} - \tilde{\eta}_0]^{2/3} \sum_{j=0}^{\infty} \hat{A}_j [\tilde{\eta} - \tilde{\eta}_0]^{j/3} \tag{6.24}$$

is consistent with (6.17). We therefore proceeded to verify that the form (6.24) is correct, determine the location of each such singular point $\tilde{\eta}_0$ and find out if there are other forms of singularities admitted by (6.17).

As suggested originally, by Kruskal & Segur (1986) for another third order nonlinear ODE, a convenient numerical method to calculate such solutions will be to march in along the positive real axis from a large distance by using the asymptotic behavior (6.19) for $\tilde{F}$, $\tilde{F}'$ and $\tilde{F}''$. This ensures that we have effectively set the coefficients of the two possible exponential terms to zero for $Arg\ \tilde{\eta}$ in the interval $(-2\pi/9, 2\ \pi/9)$, since otherwise on the real positive axis (a Stokes line), each of the two exponential exponential contributions would have dominated some algebraic terms that are included in the three term asymptotic expansion (6.19). The third degree of freedom for the third order ODE (6.17) has been utilized in demanding (6.18) rather than the more general condition $\tilde{F}\ \sim$ constant $\tilde{\eta}^{-1/2}$, which for arbitrary constant is a possible leading order asymptotic behavior for a solution to (6.17). We did not use a starting value of $\tilde{\eta}$ exceeding 15 to avoid the expected stiffness of the differential equation for large $\tilde{\eta}$. However, for a starting $\tilde{\eta}$ exceeding 10, we verified that the computed solution $\tilde{F}$ was independent of starting $\tilde{\eta}$ to 10 digits. A Runge-Kutte solver that automatically adapts the step size to control errors was to calculate numerical solution for any given $\tilde{\eta}$ on the real axis for $\tilde{\eta}$ in the interval $(-5, 15)$. This numerical integration showed no sign of singularities on the real axis on this interval and $\tilde{F}$ was always real upto machine precision. We then used a procedure quite similar to one used earlier for computing singularities in a different context (Tanveer, 1991) We took round

trip paths over anti-clockwise closed contours C in the $\tilde{\eta}$ plane and numerically evaluated the values of

$$\frac{1}{4\pi i}\oint_C d\tilde{\eta}\frac{\tilde{F}'}{F} \tag{6.25}$$

$$\frac{1}{4\pi i}\oint_C d\tilde{\eta}\;\tilde{\eta}\;\frac{\tilde{F}'}{F} \tag{6.26}$$

and found it was zero to numerical accuracy most of the times, suggesting that there are no singularities inside such contours C. Further, it was checked that $\tilde{F}$ returned to the same value. When this contour integration answers were nonzero for any contour, we also found that $\tilde{F}$ during those times did not return to the same value. So we suspected that there was one or more singularities inside our contour. In that case, we went around that contour C anti-clockwise three times and evaluated each of the integrals (6.25) and (6.26). Then, we either found that (6.25) gave us a value of 1 (within numerical accuracy) and during those times $\tilde{F}$ returned to the same value after three rotations; or at other times the value of (6.25) was not unity and during those times $\tilde{F}$ did not return to the same value even after three rotations. In the first case, we concluded that there was one singularity of the form (6.24) at a location $\tilde{\eta}_0$ given by the value of the integral in (6.26) (remembering, offcourse, that C is traversed thrice). Such a value was consistent with the location of the actual contour C, regardless of its size, which we varied. In the second case, we concluded that the contour included multiple singularity (which need not be on the same Riemann sheet). In such cases, we decreased the size of the contour C, until we found that the value of (6.25) on a thrice traversed contour was unity.

With our numerical search for $\tilde{\eta}$ in the region $-5 \le Re\,\tilde{\eta} \le 5$, $-7.5 \le Im\,\tilde{\eta} \le 7.5$, we were only able to find three pairs of singularities located at

$$\tilde{\eta}_0 = 0.29349 \pm 3.7586\,i, \quad 0.660998 \pm 5.61650\,i, \text{ and.. } 0.92209 \pm 6.9174\,i \tag{6.27}$$

It was difficult to get reliable values of the integrals in (6.25) and (6.26) for larger values of $\tilde{\eta}$ far away from the real axis because of the effect of exponentially growing terms. This led to restrictions on the zone where reliable singularity search could be performed. One curious observation about the roots in (6.27) are that they are close to the Stokes line $Arg\;\tilde{\eta} = \pm 4\,\pi/9$ that limit the validity of the asymptotic expansion (6.19) and get closer as the roots are further out from the origin. Now, a divergent asymptotic series for $\tilde{F}$ valid in some sector of the complex plane has encoded in it information about the asymptotic behavior for other sectors where the asymptotic series is invalid due to its subdominance to transcendental terms that now become large. However, the form of the transcendental correction when $Arg\;\tilde{\eta}$ is just beyond $4\pi/9$ is given the one in (6.20). Now notice that

the singularities in (6.27) are at reasonably large distances from the origin, especially the latter ones. We argue that a singularity far out from the origin ought to be located close to a point $\tilde{\eta}_g$, where

$$\tilde{\eta}_g^{-1/2} + C_1 \tilde{\eta}_g^{-13/8} e^{i\sqrt{32/729}\tilde{\eta}_g^{9/4}} = 0 \tag{6.28}$$

Offcourse, the asymptotic series (6.20) is itself valid in the immediate vicinity of such a point $\tilde{\eta}_g$, as can be expected since $\tilde{F}$ satisfying (6.17) does not admit a simple zero. However, we make the ansatz that the modification of (6.20) occurs over a small local scale when $\tilde{\eta}_0$ is large and so $\tilde{\eta}_g$ is close to $\tilde{\eta}_0$. This is admittedly a very heuristic argument; however, we now see the ramifications when this is assumed correct. Consider the zeros of (6.28) near the Stokes line $Arg\ \tilde{\eta} = 4\pi/9$. We put

$$\tilde{\eta}_g = r\ e^{i4\pi/9 + i\epsilon} \tag{6.29}$$

and carry out a perturbation for large r to find that

$$\epsilon \sim \frac{4}{9\sqrt{s_0} r^{9/4}} \left[\frac{9}{8}\ ln\ r - ln\ |C_1| \right] \tag{6.30}$$

$$r = \frac{1}{s_0^{2/9}} [2\ k\ \pi - \frac{3\ \pi}{4}]^{4/9} \tag{6.31}$$

Computing (6.31) for $k = 1,\ 2,\ 3$ gives us:

$$r = 3.6788, \quad 5.6252, \text{and} \quad 6.961618 \tag{6.32}$$

which compares well with the actual values of $|\tilde{\eta}_0|$ for the roots given by (6.27):

$$3.77004, \quad 5.65526.., \text{and} \quad 6.97858 \tag{6.33}$$

The agreement between (6.32) and (6.33) gets better for larger singularity distances from the origin, as can be expected if our heuristic argument is correct. Now, lets compare the value of ϵ for $k = 1, 2, 3$, with what is observed to be the for deviation of $Arg\ \tilde{\eta}_0$ from $4\pi/9$ for the roots in (6.27) in the upper half plane. We cannot directly apply the asymptotic relation (6.31) accurately since $|C_1|$ given in (6.21) is only known interms of a real posive number b (introduced in section 5) that we have never computed. Leaving out the $|C_1|$ term in (6.30) would have been allright as a leading order approximation if r was sufficiently large so that $|ln\ r| >> 1$. But this is not the case. We circumvent this difficulty, by estimating $|C_1|$ by using the last value in (6.33) and using it in (6.30) to determine $|C_1|$. This computed $|C_1|$ is then used in (6.30) for $k = 1,\ 2$ to give us:

$$\epsilon = 0.095909, \quad 0.05768 \tag{6.34}$$

compared to 0.096606 and 0.05738, respectively for $Arg\ \tilde{\eta}_0\ -\ 4\pi/9$ for the first two roots quoted in (6.27) (only those in the upper half plane). Again, with the asymptotic argument results is better the further out the singularity is. Thus, it appears that our heuristic argument about the infinite set of singularities along the Stokes line $4\pi/9$ may be correct. If so, their radial positions approach those given by (6.31) for to positive integral values of k in (6.31) (the agreement getting better for larger k), whose Arg values approach $4\pi/9$ from above with the deviation ϵ approaching that predicted by (6.31). We checked the $k = 4$ case with direct numerical computation which yielded a singular point at $\tilde{\eta}_0 = (1.128, 7.966)$, very close to what is predicted. Clearly, from symmetry of the problem about the positive real axis, on the basis of our heuristic argument, there will be an infinite number of roots close to the Stokes line $Arg\ \tilde{\eta}_0 = -4\pi/9$, with complex conjugate locations to the ones determined above.

Now let's examine the consequence of singularities in the similarity solution. Going back to the relation (6.13), (6.15) and (6.16) it is clear that for $\tau << 1$,

$$H(\xi,\tau) \sim -\tau^{-7/27} [\frac{\tilde{\eta}_0}{2}]^{1/3} [\xi - \xi_p(\tau)]^{2/3} \tag{6.35}$$

for

$$|\xi - \xi_p(\tau)| << \tau^{2/9} \tag{6.36}$$

where

$$\xi_p(\tau) = \tau - \tilde{\eta}_0 \tau^{2/9} \tag{6.37}$$

Thus initally each singularity coincides with $\xi = 0$, i.e. $\zeta_0(0)$. However, they move away from it for $\tau > 0$. Now for any $\tau > 0$, assuming that there are infinite number of singularities that are arbitrarily far out in the $\tilde{\eta}$ plane as indicated above that will approach the Stokes lines that limit the validity of the asymptotic expansion (6.19) (and hence inner-outer matching). These singular points located in $|\xi| >> 1$ are expected to alter the global Stokes line structure for ζ outside the inner region around $\zeta_0(t)$. It is unclear, what this affect will be. However, we proceed further with the assumption that that the global Stokes line structure is such that local matching for $Arg\ \tilde{\eta}$ (and hence $Arg\ \xi$ for large ξ) in the interval $(-4\pi/9,\ 4\pi/9)$ guarantees that transcendentally large terms in $\mathcal{B}$ are avoided on $|\zeta| = 1$ thereby ensuring the validity of (1.11).

6b. The case of $t = O(\mathcal{B}^{2/7})$

In this case τ, as defined in (6.2), is order unity. The similarity solution (6.13) becomes invalid and one has to look at other ways of solving the P.D.E. (6.10). Our primary interest in (6.10) is to see if solutions exist that satisfy the boundary condition (6.11) over some

sector of the complex ξ plane, if for early times such a solution can be matched to the early time similarity solution (6.13) and what singularities are present in such solutions.

We assume that the only singularities of (6.6) are those created at $\tau = 0$, though located at a different position that is determined from (4.12). Note that near a singularity, the ξ term has little importance and therefore the equation in an approximate sense reduces to a Harry-Dym equation (with the independent variables $\xi - \tau$ and τ) for which it might be expected that the only moving singularities are of -4/3 rd type for G_ξ. With this assumption, we expect an infinite number of singularities moving away from $\xi = 0$. Now consider the question of inner-outer matching.

What is necessary is to ensure that boundary conditions (6.11) can be satisfied for sufficiently large range of $Arg\ \xi$. For that purpose, we need to look for other possible behavior of H aside from terms that are algebraic in ξ that occur in systematic dominant balance arguments applied to (6.10) and (6.11). This can be found by linearizing (6.10) about the asymptotic behavior (6.11) and looking for solutions $\tilde{H}$ to the associated homogeneous equation, which in this case turns out to be:

$$\xi^{3/2}\left[\tilde{H}_\tau + \tilde{H}_\xi\right] - \tilde{H}_{\xi\xi\xi} = 0 \tag{6.38}$$

We impose the condition that

$$\tilde{H}(\xi, 0) = 0 \tag{6.39}$$

for any $\xi \neq 0$, as it appears appropriate beyond the immediate scale where the similarity solution of section 6a is relevant. Also, as $\xi \rightarrow \infty$ for certain range of arguments that is unspecified at this time,

$$\tilde{H}(\xi, \tau) \rightarrow 0 \tag{6.40}$$

Such a solution can be found in the form

$$\tilde{H} = e^W \tag{6.41}$$

The leading order WKB type behavior of W is given by W_0 that satisfies

$$\xi^{3/2}\left[W_{0\tau} + W_{0\xi}\right] - W_{0\xi}^3 = 0 \tag{6.42}$$

Appropriate solution to this which matches with the form of the exponential in (6.20) and (6.22) (after appropriate change of variables) for $\tau << 1$ is is

$$W_0 = \pm \frac{i\sqrt{32}(\xi - \tau)^{9/4}}{\tau^{1/2}9^{3/2}} \tag{6.43}$$

The next order correction to W that matches with the form of the algebraic prefactors in (6.20) and (6.22) (after appropriate change of variables) is

$$W_1 = \frac{1}{4} ln \left[\frac{\tau}{(\xi - \tau)^{9/2}} \right] \tag{6.44}$$

One can clearly see that the form of the leading order transcendental corrections $e^{W_0 + W_1}$ agrees with with the form for $\tau << 1$, as reflected in (6.20) and (6.22).

¿From the (6.41) and (6.43), it is clear that with a choice of positive sign, $\tilde{H}$ is transcendentally small for large ξ when $Arg\ \xi$ is in the interval $(0, \frac{4}{9}\pi)$ and with the minus sign choice $\tilde{H}$ transcendentally large for large ξ in this sector but transcendentally small when $Arg\ \xi$ is in the adjoining interval $(-\frac{4}{9}\pi, 0)$. Now, for a general partial differential equation with third order spatial derivative, one can expect to be able to specify three independent boundary conditions. For (6.10), one degree of freedom has been used in requiring (6.11) rather than the more general asymptotic condition $H \sim [\text{constant } (\xi - \tau) + \tau]^{-1/2}$. The other two degrees of freedom can be used by ensuring that the the coefficients of $\frac{\tau^{1/4}}{(\xi-\tau)^{13/8}} e^{i \frac{\sqrt{32}(\xi-\tau)^{9/4}}{\tau^{1/2} 9^{3/2}}}$ for $Arg\ \xi$ in $(-\frac{4}{9}\pi, 0]$ and and coefficients of $\frac{\tau^{1/4}}{(\xi-\tau)^{13/8}} e^{-i \frac{\sqrt{32}(\xi-\tau)^{9/4}}{\tau^{1/2} 9^{3/2}}}$ for $Arg\ \xi$ in $[0, \frac{4}{9}\pi)$ are all zero. For $\tau << 1$, we have seen in terms of compuation of a similarity solution, that these conditions are appropriate.

This would mean that it is possible to satisfy the asymptotic boundary condition (6.11) for $Arg\ \xi$ in the open interval $(-\frac{4\pi}{9}, \frac{4\pi}{9})$. There is nothing special about this range of argument of ξ. Clearly, instead of this range, we can choose to ensure boundary condition (6.11) for other range of arguments like $(0, \frac{8\pi}{9})$), etc. The appropriate choice will depend on $q_{7_0}(\zeta_0(t), t)$, etc. that relate $Arg\ (\zeta - \zeta_0(t))$ to $Arg\ \xi$ through (6.1). Certainly $\frac{8}{9}\pi$ is smaller than π. Thus the proposed sufficient condition (section 5) for inner-outer matching for the nearest singularity is not applicable here. However, if we are to assume that there are an infinite set of $-4/3$ singularities (as our arguments indicate) that change the global Stokes lines in such a way that the matching in an appropriate $8\pi/9$ ensures the validity of (1.11) in the physical domain, then we have a consistency in inner-outer matching. We cannot ascertain this at this stage without studying the global Stokes line structure, but proceed further on the assumption that this is the case.

Further in the sectors where the asymptotic expansion (6.11) is valid, an examination of the two term asymptotic relation for $G_\xi = H^{-2}$, obtained directly from (6.6) and using by using the leading order relation (6.7) implies that

$$G_\xi = \xi - \frac{3}{2} \xi^{-5/2} + \frac{3}{2} (\xi - \tau)^{-5/2} \ldots \tag{6.45}$$

which for $\xi \rightarrow \infty$ which for $|\tau| << |\xi|$ matches to (4.10), as it must for a consistent matching.

6c. $\mathcal{B}^{-2/7}\ t\ >>\ 1$

In this case, $|\tau|\ >>\ 1$. For $|\tau|\ >> |\xi|\ >> 1$, it is clear that the τ dependence in the asymptotic behavior will disappear at each order (though not uniformly). Thus, for large τ the inner equation (6.6) for τ independent solution becomes:

$$0\ =\ -G_\xi\ +\ \xi\ -\ 2\left[G_\xi{}^{-1/2}\right]_{\xi\xi} \tag{6.46}$$

On substituting $G_\xi^{-1/2}\ =\ H$, we get the inner equation obtained in the steady state analysis of Combescot et al (1988) for the relative finger width $\lambda\ <\ \frac{1}{2}$. Via Borrel summation, they showed that it is possible to have solutions so that

$$G_\xi\ \rightarrow \xi \tag{6.47}$$

for $Arg\ \xi$ in $[0, \frac{6\pi}{7})$. However, in the steady state problem, it is necessary to have (6.24) valid for $Arg\ \xi$ in $[-\frac{\pi}{2},\ \frac{\pi}{2}]$, or equivalently (for symmetric steady finger with a smooth tip) require $Arg\ G_\xi\ =\ 0$ when $Arg\ \xi\ =\ 0$ for sufficiently large ξ. These were found not to be valid and so a steady finger was not possible in that case. In our time dependent problem with $\mathcal{B}$ independent initial conditions, we have a different picture. Assume at each instant of time, a zero $\zeta_0(t)$ does not move towards the physical domain as fast as the corresponding daughter singularity $\zeta_{s1}(t)$, as is true for the Saffman family of solutions. Then it is enough to require that the solution G_ξ satisfy (6.24) for $Arg\ \xi$ in $[0,\ \pi/2]$ and a separate solution (not an analytical continuation) of (6.46) satisfy (6.47) for $Arg\ \xi$ in $[-\pi/2, 0]$. These two solutions need not be identical since we can have branch cut connecting $\zeta_0(t)$ and $1\ <<\ \xi\ <<\ \tau\ << \mathcal{B}^{-2/7}$, this corresponds to a cut between $\xi\ =\ 0$ and $\xi\ =\ \tau$. Thus, for $\tau\ >>\ \xi\ \ >> 1$ a matching of the inner solution is possible to the outer solution of the form (4.1). Clearly, when $\xi\ >>\ \tau\ >>\ 1$, the above analysis becomes invalid. This is not unexpected as there is a singularity of the outer perturbation expansion (1.11) at $\xi\ =\ \tau$, corresponding to $\zeta\ =\ \zeta_{s_1}(t)$ that moves away from the inner zone around $\zeta\ =\ \zeta_0(t)$. It then becomes necessary to introduce an inner-neighborhood around $\zeta = \zeta_{s_1}(t)$. We do that in the next section for $t\ =\ O(1)$. The result remains valid for $t\ >>\ \mathcal{B}^{2/7}$ and so there is no necessity to go through the analysis for $\tau\ >>\ 1$ for $\xi - \tau\ =\ O(1)$.

Further, when $\tau\ >>\ 1$, each of the -4/3 rd singularities $\zeta_p(t)$ that was born at $\zeta_0(0)$ move away from the inner region around $\zeta\ =\ \zeta_0(t)$. This will be dealt with in the following section as well.

7. Inner scales around each created singularity for $t\ =\ O(1)$

From, the result (6.35) and the relation of H, ξ and τ to z_ζ, ζ and τ that can be found from (6.1)-(6.3) and (6.9), it follows that for $t\ <<\ \mathcal{B}^{2/7}$, $\tilde{A}_0(t)$ defined in (4.11) scales as

$[\mathcal{B}t]^{14/27}$ (Note the scaling does on $\tilde{\eta}_0$ associated with that particular singularity). We make the ansatz that when $\zeta_{s_1}(t)$, the newly created singular point of the outer perturbation expansion as noted in section 4, is not in the immediate vicinity of $|\zeta| = 1$, $\tilde{A}_0(t)$ scales as $\mathcal{B}^{14/27}$ for $t = O(1)$ as well. When $\zeta_{s_1}(t)$ becomes very close $|\zeta| = 1$, the inner equations themselves become global integro-differential equations (as will be seen in section 8 for special cases) and in that case, we do not expect the $\mathcal{B}^{14/27}$ scaling to survive. On integration of (4.12), we get

$$\zeta_p(t) = -\int_0^t q_1(\zeta_p(t),t)dt' - \frac{4}{9}\mathcal{B}^{2/9}\int_0^t dt' q_7(\zeta_p(t'),t)\,[\mathcal{B}^{-14/27}\,\tilde{A}_0(t')\,]^{-3/2} \tag{7.1}$$

¿From section 4, $\dot{\zeta}_{s_1}(t) = -q_{1_0}(\zeta_{s_1}(t),\, t)$. Since initially $\zeta_p(0) = \zeta_{s_1}(0) = \zeta_0(0)$, it follows from (7.1) and our ansatz about the scaling of $\tilde{A}_0$ that

$$\zeta_p(t) = \zeta_{s_1}(t) + O(\mathcal{B}^{2/9}) \tag{7.2}$$

Now, consider first the inner region around $\zeta = \zeta_{s_1}(t)$. It turns out that this does not include or overlap with the inner region around any of the $\zeta_p(t)$ Clearly, as pointed out in the last section, for $t >> O(\mathcal{B}^{2/7})$, a singular point ζ_{s1} of the outer perturbation expansion (4.1), initially coinciding with ζ_0 , moves towards the physical domain. We now introduce the inner equations near $\zeta_{s1}(t)$ where this apparent singularity in the outer equation is smoothed out. The appropriate scaled variables in this case are:

$$\zeta - \zeta_{s_1}(t) = \mathcal{B}^{1/3}\; e^{-\int_0^t dt'\, q_{1_{0\zeta}}(\zeta_{s_1}(t'),t')}\,\tilde{\nu} \tag{7.3}$$

$$z_\zeta = z_{0\zeta} + \mathcal{B}^{1/6}\,A_1(0)\; e^{\int_0^t dt'\, q_{1_{0\zeta}}(\zeta_{s_1}(t'),t')}\,\tilde{H}(\tilde{\nu},\,\tau) \tag{7.4}$$

where

$$\tau = \int_0^t dt'\,\frac{q_{7_0}(\zeta_{s_1}(t'),t')}{z_{0\zeta}^{3/2}\,(\zeta_{s_1}(t'),\,t')}\; e^{3\int_0^{t'} dt_1\, q_{1_{0\zeta}}(\zeta_{s_1}(t_1),t_1)} \tag{7.5}$$

Then, to the leading order, we find

$$\tilde{H}_\tau = \tilde{H}_{\tilde{\nu}\tilde{\nu}\tilde{\nu}} \tag{7.6}$$

The appropriate solution to this that matches with the singularity given by (4.8) is:

$$\tilde{H} = \frac{1}{\tau^{5/6}}\,\tilde{F}(\frac{\tilde{\nu}}{\tau^{1/3}}) \tag{7.7}$$

where $\tilde{F}$ satisfies the ordinary differential equation:

$$-\frac{5}{6}\,\tilde{F} - \frac{1}{3}\,\tilde{\eta}\,\tilde{F}'(\tilde{\eta}) = \tilde{F}_{\tilde{\eta}\tilde{\eta}\tilde{\eta}} \tag{7.8}$$

It is clear that for large $\tilde{\eta}$, one possible behavior is given by

$$\tilde{F} \sim \tilde{\eta}^{-5/2} \tag{7.9}$$

and this matches with (4.8). In order to ascertain that (7.9) holds over a big enough sector of the complex plane, we consider WKB solution to (7.10), which clearly can be a linear combination of the two asymptotic behavior:

$$\tilde{\eta}^{1/2} \; e^{\pm i \; \frac{2}{3^{3/2}} \; \tilde{\eta}^{3/2}} \tag{7.11}$$

We can ensure the asymptotic behavior (7.9) for $Arg \; \tilde{\eta}$, i.e. $Arg \; \xi$ in $(-\frac{2}{3}\frac{\pi}{}, \frac{2\pi}{3})$ which is a combination of two Stokes sectors. In that case, for $Arg \; \tilde{\eta}$ in $(0, \frac{2\pi}{3})$, including the leading order transcendental correction, the asymptotic behavior for $\tilde{F}$ will be

$$\tilde{F} \sim \tilde{\eta}^{-5/2} + .. + D_1 \; \tilde{\eta}^{1/2} \; e^{i \; \frac{2}{3^{3/2}} \; \tilde{\eta}^{3/2}} \tag{7.12}$$

For $Arg \; \tilde{\eta}$ in $(-\frac{2\pi}{3} \; , \; 0)$

$$\tilde{F} \sim \tilde{\eta}^{-5/2} + .. + D_2 \; \tilde{\eta}^{1/2} \; e^{-i \; \frac{2}{3^{3/2}} \; \tilde{\eta}^{3/2}} \tag{7.13}$$

where D_1 and D_2 are uniquely determined order unity constants. Since the sector of matching is bigger that π , then according to suggested matching principle in section 5, an inner-outer matching is possible in this case.

Now consider the inner-zone around each singularity $\zeta_p(t)$. It is appropriate to introduce inner variables:

$$\zeta - \zeta_p(t) = \mathcal{B}^{7/18} \; \tilde{\chi} \tag{7.14}$$

$$z_(\zeta, t) \; = \; \frac{1}{\tilde{H}^2(\tilde{\chi})} \tag{7.15}$$

Then plugging it into (2.17), the leading order equation equation is

$$\frac{\tilde{H}_{\tilde{\chi}}}{\tilde{H}^3} \; + \; M(t) \; \tilde{H}_{\tilde{\chi}\tilde{\chi}\tilde{\chi}} \; = \; 0 \tag{7.16}$$

where

$$M(t) \; = \; \frac{\mathcal{B}^{2/9} q_{7_{0_\zeta}}(\zeta_p(t), t)}{\dot{\zeta}_p + q_{1_0}(\zeta_p(t), t)} \tag{7.17}$$

In order to match with the outer solution, we must require that as $\tilde{\chi} \; \rightarrow \; \infty$ in certain sectors (to be specified) that correspond to certain directions towards the physical ζ domain,

$$\tilde{H} \; \rightarrow \; \tilde{H}_\infty \tag{7.18}$$

where

$$\tilde{H}_\infty = [z_{0_\zeta}((\zeta_p(t), t)\,]^{-1/2} \tag{7.19}$$

Solution to the ordinary differential equation (7.16), where $M(t)$ is merely a parameter, can be obtained in closed form (in terms of elliptic functions) that ensures the validity of the matching (7.18) for $Arg\ \tilde{\chi}$ in $((2k-1)\ \pi,\ (2k+1)\ \pi)$ for any specific choice of integer k. A choice of integer k clearly exists so that inner-outer matching is possible for $Arg\ (1 - \zeta/\zeta_p(t))$ in $(-\pi/2,\ \pi/2)$.

8. Final stages for specific type of singularities of z_0

In the previous section, our analytical evidence suggests a very rich dynamics that involve not only the alteration of initial singularities but also the creation of new singular points. As mentioned earlier, all singularities approach the physical domain $|\zeta| = 1$, as to the leading order, their location evolves according to the relation $\dot{\zeta} = -q_{1_0}(\zeta(t), t)$. Thus eventually each of $\zeta_s(t)$, $\zeta_{s_1}(t)$ and $\zeta_p(t)$ will be arbitrarily close to $|\zeta| = 1$. In that case the leading order approximation $q_1 \sim= q_{1_0}$, $q_7 = q_{7_0}$, etc. that were used in deriving the leading order inner equations in sections 5, 6 and 7 become invalid. It is clear then that we need to look at the modifications of the global integral terms q_1, q_7, etc, created by the presence of singularities that are very close to $|\zeta| = 1$ in order to study this late stages. This is a rather involved study by itself.

On the other hand, unless we study this, our findings in the previous sections will remain of uncertain relevance to the physically interesting localized distortions of the Hele-Shaw interface that occur in the presence of singularities very close to $|\zeta| = 1$. In this section, we undertake the task of deriving appropriate inner-equations when a singularity $\zeta_s(t)$ of z_{0_ζ} of the form $z_{0\zeta} \sim B_0(t)\ (\zeta - \zeta_s(t)^{-\beta}$ approaches the physical domain arbitrarily closely in the restricted case $\frac{1}{2} < \beta < 1$. Our main purpose will be to show that z_ζ scales as some inverse power of $\mathcal{B}$ over this localized scale implying that the approach of the singularities of this kind of the zero surface tension solution does indeed correspond to large distortions of the physical interface, which are smoothed over a small local scale by surface tension effects. We expect similar effects by the approach of other kinds of singularities not considered here (including the -4/3 types created by surface tension effects).

In this case, equation (2.17) is not a convenient starting point, as many of the terms become of the same order. We now consider $\zeta = e^{i\nu}$ to be a point on $|\zeta| = 1$ corresponding to the physical interface. We work directly with (2.1) and (2.11) which remains valid on $|\zeta| = 1$ provided the integrals I_1 and I_4 are taken in the limit of ζ approaching the circle from inside. We introduce the inner-scales:

$$-i(1 - e^{i\nu}/\zeta_s(t)) = \mathcal{B}_1^p\ \xi \tag{8.1}$$

$$z_\zeta = \mathcal{B}^{-\beta\, p_1} H(\xi, t) \tag{8.2}$$

The scaling of (6.2) is appropriate in order to match with the outer-behavior $z_\zeta \sim B_0(t)\,(\zeta - \zeta_s(t))^{-\beta}$ Further we introduce

$$\omega = \mathcal{B}^{1-p_1(1-\beta)}\, \Omega(\xi, t) \tag{8.3}$$

We introduce

$$\zeta_s(t) = R(t)\, e^{i\nu_s(t)} \tag{8.4}$$

and will define

$$\rho(t) = (R-1)\, \mathcal{B}^{-p_1} \tag{8.5}$$

For $\rho >> 1$, we recover the local inner-equations studied in section 5. So, we will assume $\rho = O(1)$. Then, on substituting the above relation into (2.1), we find that to the leading order,

$$\Omega(\xi, t) = \frac{1}{\pi i} \int_{-i\rho-\infty}^{-i\rho+\infty} d\xi' \, \frac{1}{\xi - \xi'} \, \frac{1}{|H(\xi', t)|} \, Im \left[\frac{H_\xi(\xi', t)}{H(\xi', t)} \right] + \frac{1}{|H(\xi, t)|} \, Im \left[\frac{H_\xi(\xi, t)}{H(\xi, t)} \right] \tag{8.6}$$

where the integral above is in the principal value sense. Now, consider the kinematic condition (2.11). First, it is useful to break up $I_4(\zeta, t)$ as

$$I_4(\zeta, t) = I_4^0(\zeta, t) + I_4^R(\zeta, t) \tag{8.7}$$

where

$$I_4^0(e^{i\nu}, t) = -\frac{1}{\pi^2} \int_0^{2\pi} d\nu' \frac{e^{i\nu} + e^{i\nu'}}{e^{i\nu'} - e^{i\nu}} \, \frac{1}{|z_\zeta(e^{i\nu'}, t)|^2} - \frac{2}{\pi |z_\zeta(e^{i\nu'}, t)|^2} \tag{8.8}$$

$$I_4^R(e^{i\nu}, t) = \frac{1}{2\pi} \int_0^{2\pi} d\nu' \frac{e^{i\nu} + e^{i\nu'}}{e^{i\nu'} - e^{i\nu}} \, \frac{Re\,[e^{i\nu'}\, \omega_\zeta(e^{i\nu'}, t)]}{|z_\zeta(e^{i\nu'}, t)|^2} + \frac{Re\,[e^{i\nu}\, \omega_\zeta(e^{i\nu}, t)]}{|z_\zeta(e^{i\nu}, t)|^2} \tag{8.9}$$

It is not difficult to establish that

$$\begin{aligned} I_4^0 \sim & -\frac{1}{\pi^2 i} \int_0^{2\pi} d\nu' cot \, \frac{\nu' - \nu_s(t)}{2} \, \frac{1}{|z_{0\zeta}(e^{i\nu}, t)|^2} \\ & + \frac{\mathcal{B}^{p_1}(\xi + i\rho(t))}{\pi^2 i} \int_0^{2\pi} d\nu' cot \, \frac{\nu' - \nu_s(t)}{2} \, \frac{\partial}{\partial \nu'} \, \frac{1}{|z_{0\zeta}(e^{i\nu'}, t)|^2} + o(\mathcal{B}^{p_1}) \end{aligned} \tag{8.10}$$

Again

$$I_4^R \sim -\mathcal{B}^{1-2p_1+3\beta p_1} \left[\frac{1}{\pi i} \int_{-i\rho-\infty}^{-i\rho+\infty} \frac{d\xi'}{\xi - \xi'} \, \frac{Im\ \Omega_\xi(\xi', t)}{|H(\xi', t)|^2} + \frac{Im\ \Omega_\xi(\xi, t)}{|H(\xi, t)|^2} \right] \tag{8.11}$$

Using the scalings (8.1), (8.2) and substituting (8.10) and (8.11) into (8.7) using the relation for $\dot{\zeta}_s/\zeta_s$ derived in section 3 and the relation (3.44), we find that it is necessary to choose

$$p_1 = \frac{1}{3(1-\beta)} \tag{8.12}$$

to get a leading order inner-equation that includes the curvature and which can match to the outer solution. Obviously, this scaling is invalid for $\beta \geq 1$, since the assumed small inner-scale is then inconsistent with (8.1). Further, the approximation (8.10) become invalid for $\beta \leq \frac{1}{2}$. With the choice in (8.12), we get (2.11) (after differentiating with respect to ζ) to reduce to

$$H_t = i\, e^{-i\nu_s(t)} \left[(I_4^{0,2}(t)\, \xi - I_4^{R,0}(\xi,t))H\right]_\xi + I_4^{0,0}(t)\, H \tag{8.13}$$

where

$$I_4^{0,0}(t) = -\frac{1}{\pi^2 i} \int_0^{2\pi} d\nu' cot\, \frac{\nu' - \nu_s(t)}{2}\, \frac{1}{|z_{0\zeta}(e^{i\nu}, t)|^2} \tag{8.14}$$

$$I_4^{0,2}(t) = +\frac{1}{\pi^2 i} \int_0^{2\pi} d\nu' cot\, \frac{\nu' - \nu_s(t)}{2}\, \frac{\partial}{\partial \nu'} \frac{1}{|z_{0\zeta}(e^{i\nu'}, t)|^2} \tag{8.15}$$

$$I_4^{R,0}(\xi,t) = \left[\frac{1}{\pi i} \int_{-i\rho-\infty}^{-i\rho+\infty} \frac{d\xi'}{\xi - \xi'}\, \frac{Im\, \Omega_\xi(\xi',t)}{|H(\xi',t)|^2} + \frac{Im\, \Omega_\xi(\xi,t)}{|H(\xi,t)|^2}\right] \tag{8.16}$$

It is then necessary to solve the system of integro differential equations (8.6) and (8.13), subject to the asymptotic behavior that

$$H(\xi,t) \sim B_0(t)[-i\zeta_s(t)]^{-\beta}\, \xi^{-\beta} \tag{8.17}$$

and for early enough times, when $\rho(t) >> 1$, we want

$$H(\xi,t) \sim B_0(t)[-i\zeta_s(t)]^{-\beta}\, \xi^{-\beta} \tag{8.18}$$

for all ξ. We have not solved these equations. Nonetheless, the scalings obtained in (8.12), together with the relations (8.1) and (8.2) immediately imply that a large localized distortion occurs in the physical interface at a point corresponding to $\zeta = e^{i\nu_s(t)}$.

Thus, we can conclude that an initial singularity of $z_\zeta(\zeta, 0)$ atleast of the type $(\zeta - \zeta_s(0))^{-\beta}$ with β in the range $(\frac{1}{2}, 1)$ initially located at $O(1)$ distance from $|\zeta| = 1$, while being smeared out and transformed by surface tension effects, will still make its presence felt on $|\zeta| = 1$ after $O(1)$ time that can be computed by calculating the trajectory of $\zeta_s(t)$ that moves according the the zero surface tension solution and eventually comes arbitrarily close to $|\zeta| = 1$.

Conclusion

We have imbedded the zero surface tension problem in a well posed problem and shown how surface tension affects singularities of certain forms. Here we only addressed the global qualitative features of the dynamics of singularities and zeros. These will be supplemented with actual numerical calculations of singularity trajectories relating it to the shape of the physical interface. We also need to account for the final stages of singularities other than the special types considered in section 8.

However, in order to completely relate the findings with physical domain initial conditions, which in general cannot be analytically continued exactly to the complex plane, one needs to address the issue of the probability density of singularities of different types and their strengths in $|\zeta| > 1$, once a function is specified in $|\zeta| \leq 1$ within a certain error (say in the Max norm sense). This does not make sense at this stage since the singularity types are not even denumerable. However, if the long time feature of the interface is not crucially dependent on the precise singularity nature, it may make sense to restrict to specific kinds of singularities. Our findings in this paper is in some sense equivalent to finding the input-output relationship of a device. However, in experiment or in direct numerical simulation (if one is possible for very small $\mathcal{B}$) what one observes is the statistical features of the output, given random inputs, the randomness coming in because one cannot control the location of singularities in the unphysical domain even with arbitrarily accurate data in the physical domain. Thus to predict the statistical features of the output, besides figuring out the input-output relationship that has been addressed to in this paper, we also need to predict the statistical features of the input. We can also expect that such statistical features of the input, combined with the understanding of the dynamics with specific input is likely to lead us to discover self-similarities of the statistical features in time that may well explain the apparently fractal nature of the interface over some range of length scales.

We can perhaps expect this mode of attack to be useful for the understanding of the time dependent features (such as side-branching) of a needle crystal, since the important element of the analytical technique is not the conformal map, but but a system of integro differential equations, which is available in that case.

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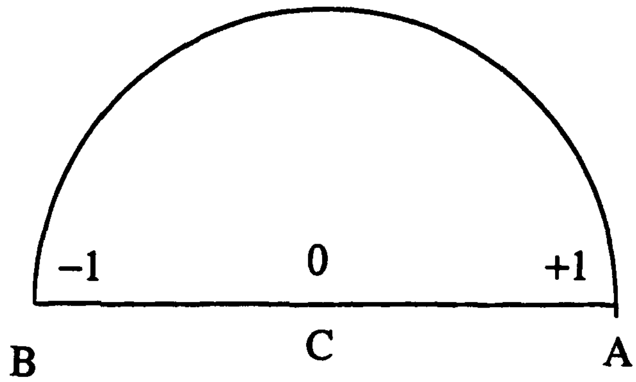
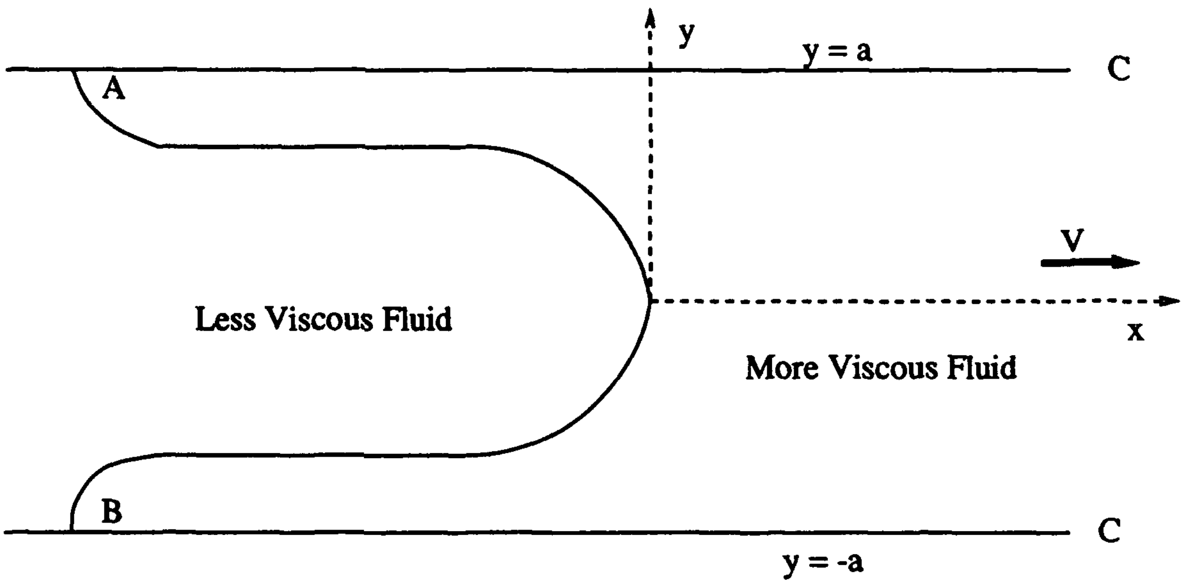
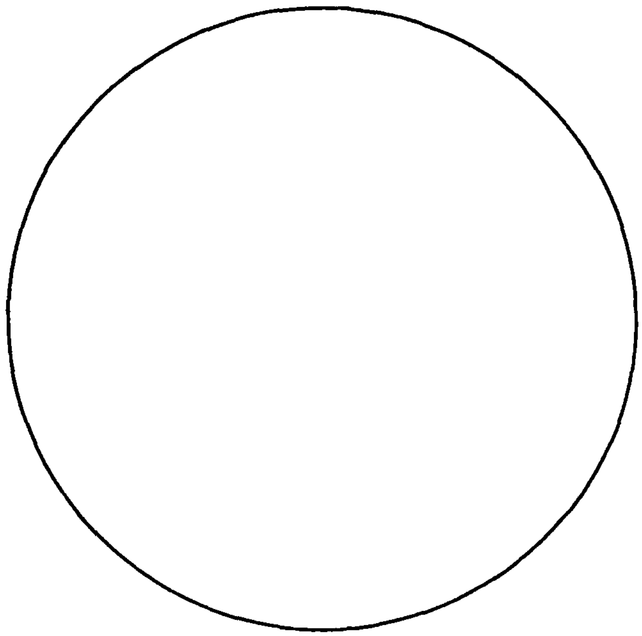


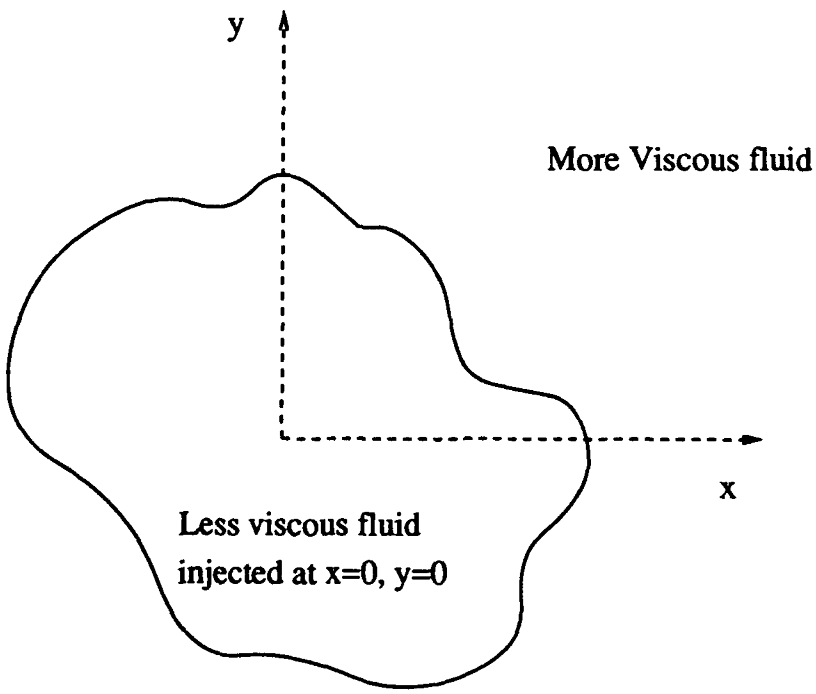
1. Fig. 1: Unit semi-circle in the ζ -plane.



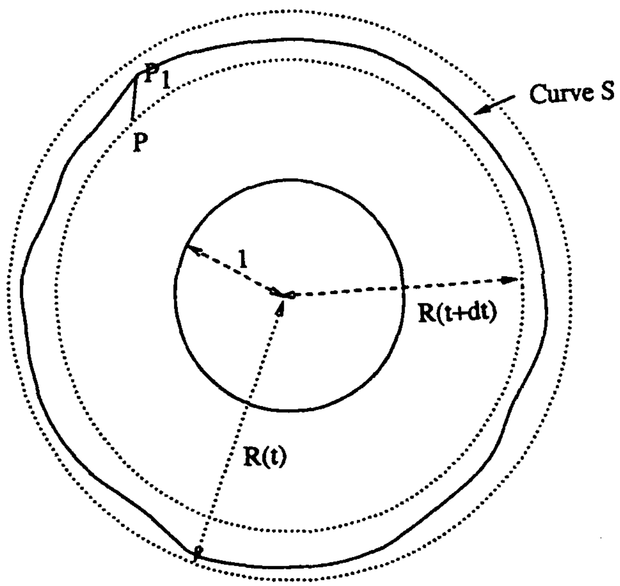
2. Fig. 2: Rectilinear Hele-Shaw flow viewed in the lateral plane.



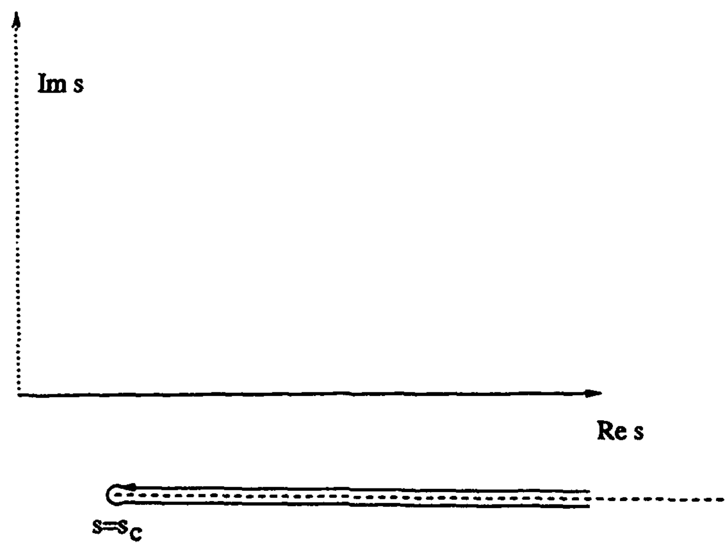
3. Fig. 3: Unit ζ circle; work plane for the radial flow.



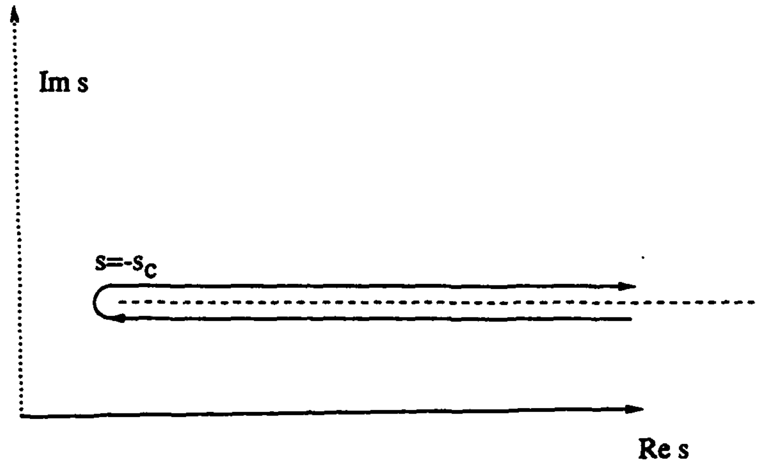
4. Fig. 4: Hele-Shaw flow in a radial geometry.



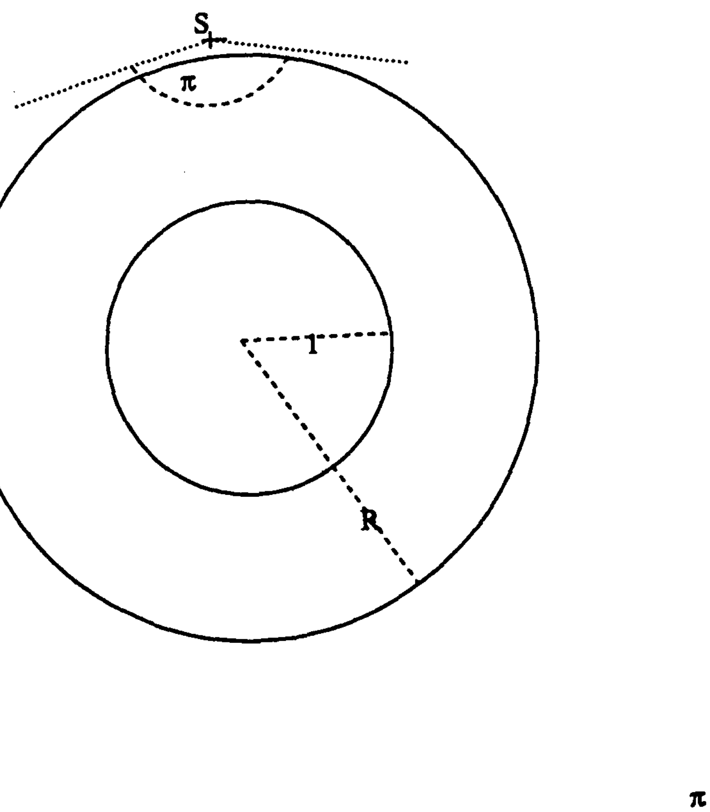
5. Fig. 5: Circles of radii $R(t)$ and $R(t + \delta t)$ in the ζ plane. Values of z on curve S determine z at time t determine z on circle of radius $R(t + \delta t)$ at time $t + \delta t$. P is a representative point on circle or radius $R(t+\delta t)$ that is influenced by point P_1 at earlier time t through characterestic as shown.



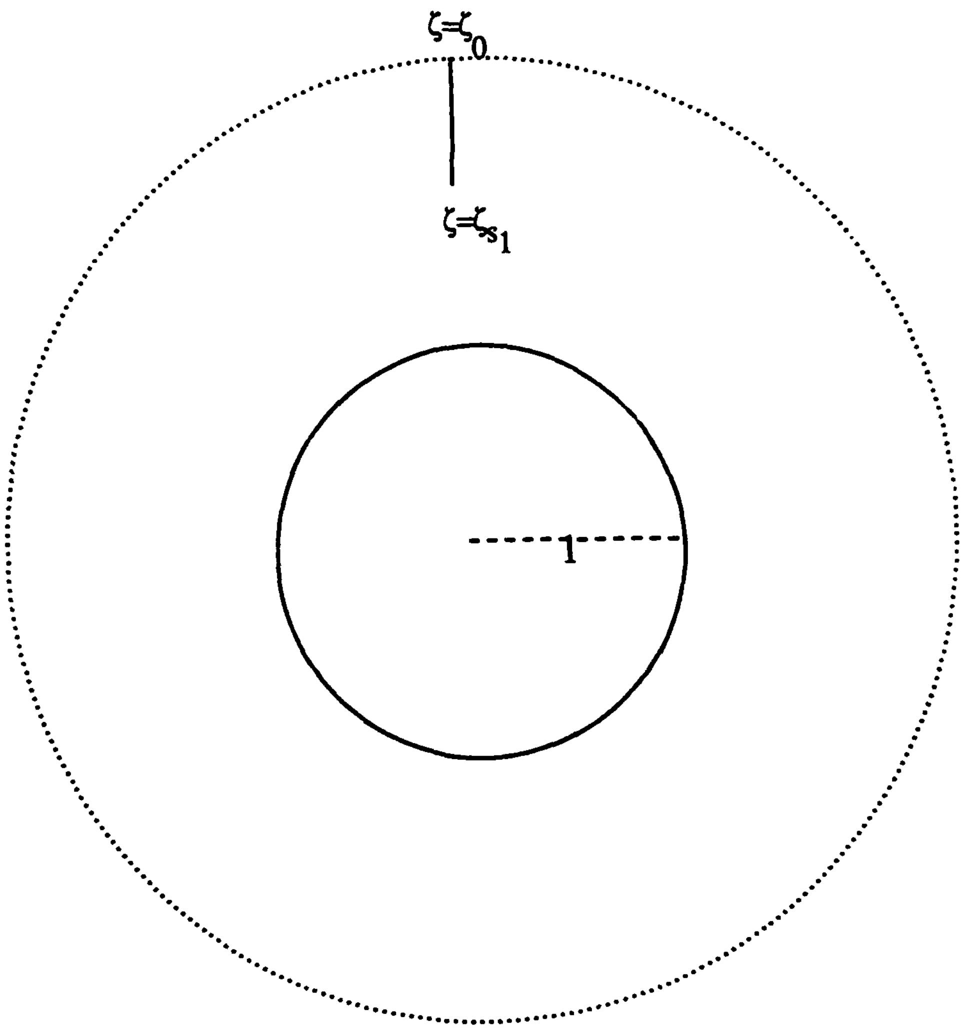
6. Fig. 6: Illustration of matching zone. Transcendentally large terms in B have to be avoided inside circle of radius R where the nearest singularity S is just beyond R . The sector of matching near S is bounded by the two dotted lines through S and subtend an angle of π .



7. Fig. 7: Deformed contour C starting at 0 along the real axis and then again along two sides of the cut at $s = s_c$ for $Arg\ \eta$ in $(\pi,\ 2\pi)$.



8. Fig. 8: Deformed contour C starting at 0 along the real axis and then again along two sides of the cut at $s = -s_c$ for $Arg\ \eta$ in $(-2\,\pi, -\,\pi)$.



9. Fig. 9: Branch cut between $\zeta = \zeta_0$ and corresponding daughter singularity $\zeta = \zeta_{s1}$ allows two different local solutions on two sides of the cut to be matched to the outer solution in the region inside the large dotted circle.

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13. ABSTRACT *(Maximum 200 words)*

For the time evolution of a Hele-Shaw interface described by a conformal map $z(\zeta, t)$ that maps a unit circle (or a semi-circle) in the ζ plane into the viscous fluid flow region in the physical z-plane, we present results on the motion of singularities outside the unit circle.

For zero surface tension, we extend earlier results to show that for any initial condition, each singularity of $z(\zeta, t)$ present initially in $|\zeta| > 1$ continually approaches the interfacial boundary $|\zeta| = 1$ without any change of form. However, depending on the singularity type, it may or may not impinge $|\zeta| = 1$ in finite time. Under some assumptions, we give analytical evidence to suggest that the ill-posed problem in the physical domain $|\zeta| \leq 1$ can be imbedded in a well-posed problem in $|\zeta| \geq 1$. We present a numerical scheme to calculate such solutions.

For each initial singularity of certain type, which in the absence of surface tension would have merely moved to a new location $\zeta_s(t)$ at time t from an initial $\zeta_s(0)$, we find an immediate transformation of the singularity structure for nonzero surface tension B; however, for $0 < B << 1$, surface tension effects on this singularity are limited to a small 'inner' neighborhood of $\zeta_s(t)$ when $t << \frac{1}{B}$. Outside the inner-region but for $|\zeta - \zeta_s(t)| << 1$, the singular behavior of z^0, the zero surface tension solution still persists for $z(\zeta, t)$. On the other hand, for each initial zero of z_ζ, which for surface tension $B = 0$ remains a zero of z^0_ζ at a location $\zeta_0(t)$ different from $\zeta_0(0)$, surface tension effects spawns new singularities that move away from $\zeta_0(t)$ and approach the physical domain $|\zeta| = 1$.

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